Imprimitive irreducible modules for finite quasisimple groups

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Abstract

Motivated by the maximal subgroup problem of the finite classical groups we begin the classification of imprimitive irreducible modules of finite quasisimple groups. A module of a group G over a field K is *imprimitive*, if it is induced from a module of a proper subgroup of G.

We obtain our strongest results when $\mathrm{char}(K) = 0$, although much of our analysis carries over into positive characteristic. If G is a finite quasisimple group of Lie type, we prove that an imprimitive irreducible KG-module is Harish-Chandra induced. This being true for $\mathrm{char}(K)$ different from the defining characteristic of G, we specialize to the case $\mathrm{char}(K) = 0$ and apply Harish-Chandra philosophy to classify irreducible Harish-Chandra induced modules in terms of Harish-Chandra series, as well as in terms of Lusztig series. One of the surprising outcome of our investigations is the fact that, asymptotically, most of the irreducible KG-modules are imprimitive in this situation.

For exceptional groups of Lie type of rank at most 2, and for sporadic groups, we list all ordinary irreducible imprimitive characters.

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CHAPTER 1

Introduction

This monograph is a contribution to the classification of the maximal subgroups of the finite classical groups and the start of the program of classifying imprimitive modules for finite quasisimple groups. The two programs are inextricably linked in a way we now explain. Suppose that we want to study the maximal subgroups of a finite classical group X whose natural module is V.

In his 1984 paper on the maximal subgroups of classical groups [**1**] Aschbacher defines eight collections of geometric subgroups $\mathcal{C}_i(X)$, $1 \leq i \leq 8$. His main theorem asserts that if L is maximal in a classical group X with natural module V, then either L is an element of $\mathcal{C}_i(X)$ for some $1 \leq i \leq 8$, or the following are true:

(1) $F^*(L)$ is quasisimple;
(2) $F^*(L)$ acts absolutely irreducibly on V;
(3) the action of $F^*(L)$ on V can not be defined over a smaller field;
(4) any bilinear, quadratic or sesquilinear form on V that is stabilized by $F^*(L)$ is also stabilized by X.

(Here, $F^*(L)$ denotes the generalized Fitting subgroup of L.) Aschbacher denotes the collection of subgroups of X which satisfy Conditions 1 through 4 above by $\mathcal{S}$.

In their recent book [**12**], Bray, Holt and Roney-Dougal have determined the maximal subgroups of the classical groups with $\dim(V) \leq 12$. This dimension is chosen to complement the work by Kleidman and Liebeck in [**69**]. In this monograph the authors determine the maximal members of each collection $\mathcal{C}_i(X)$, and for $\dim(V) > 12$ they determine those instances where a maximal member of $\mathcal{C}_i(X)$ is a maximal subgroup of X. In case a maximal member L of some $\mathcal{C}_i(X)$ fails to be maximal in X, the overgroups of L are also determined. To complete the determination of all maximal subgroups of X one needs to answer the following question. When is a member of $\mathcal{S}$ maximal in X?

We note that if $L \leq X$ is a member of $\mathcal{S}$, then so is $N_X(L)$. Hence in order to investigate the question of the maximality of L in X, we may assume that $L = N_X(L)$. Now if $L \leq \tilde{L} \leq X$, then the definition of $\mathcal{S}$

implies that $\tilde{L} \notin \mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_5 \cup \mathcal{C}_8$. If $\tilde{L} \in \mathcal{C}_4 \cup \mathcal{C}_7$ then L stabilizes a tensor product decomposition of V. Examples of such (L, V) can be found in the paper [**86**] by Magaard and Tiep, and in the papers [**5, 6, 7**] by Bessenrodt and Kleshchev. Examples of (L, V) where $\tilde{L} \in \mathcal{C}_6$ can be found in a preprint by Magaard and Tiep [**87**].

The examples (L, V) where $\tilde{L} \in \mathcal{S}$ come in several varieties depending on the isomorphism types of L, $\tilde{L}$ and X. For instance by generalizing work of Dynkin, the examples where L, $\tilde{L}$ and X are of Lie type and of the same characteristic, were classified by Seitz and Seitz-Testerman (see [**96, 100, 107**]). Liebeck, Saxl and Seitz classified the examples where L and X are of Lie type of equal charactersitic, but $\tilde{L}$ is not (see [**75**]). A list of possible examples where L and $\tilde{L}$ are of Lie type and equal characteristic but of characterisitic coprime to V was determined by Seitz in [**98**]. In her thesis [**94**] Schaeffer Fry has resolved one family of these difficult cases. In recent work Magaard, Röhrle and Testerman produced a possible list of examples where X and $\tilde{L}$ are of Lie type and of equal characteristic, and L is of Lie type but of charactersitic coprime to that of X (see [**85**]). Finalizing this list is work in progress by Magaard and Testerman. The case where $F^*(L)$ is alternating was the subject of Husen's thesis and subsequent work. Only one infinite series of examples occurs when V is the kth exterior power of the reduced permutation module W of $F^*(L)$, where $2 \leq k \leq \dim(W)/2$. In this case $X = \Omega(V)$ (if X is quasisimple) and $F^*(\tilde{L}) = \Omega(W)$. For these results see [**61, 62, 63**].

The situation to which the results of this monograph apply is when $\tilde{L} \in \mathcal{C}_2$, which implies that L acts imprimitively on V. In particular this implies that $F^*(L)$ also acts imprimitively on V. As $F^*(L)$ is quasisimple, this is precisely the situation we study here.

To introduce our results, let K be a field and G a finite group. A KG-module will always be assumed to be finitely generated with G acting from the right.

Definition 1.1. We say that a KG-module M is *imprimitive with block stabilizer* H, if and only if there are subspaces $M_1, M_2, \ldots, M_n$ with $n = [G\colon H] > 1$ such that

$$M = \bigoplus_{i=1}^{n} M_i, \tag{1.1}$$

the M_i are transitively permuted by the action of G and H is the stabilizer of M_1, i.e., $H = \{g \in G \mid M_1 g = M_1\}$.

In other words a KG-module M is imprimitive with block stabilizer H if and only if H is a proper subgroup of G and there is a KH-module M_1 such that $M \cong \mathrm{Ind}_H^G(M_1) := M_1 \otimes_{KH} KG$.

Let us now describe the principal results of this monograph in more detail. If a proper subgroup H of G is the block stabilizer of some KG-module, then so is any intermediate subgroup $H \leq L \leq G$, since induction is transitive. Thus in order to classify the imprimitive irreducible KG-modules, one may restrict attention to the maximal subgroups H of G as potential block stabilizers. We therefore assume that H is a maximal subgroup of G in the following. We also assume that K is algebraically closed.

In Chapter 2 we collect some general results from representation theory which are used to dispose of some subgroups as possible block stabilizers or to compare imprimitivity in different characteristics. We also comment on some aspects of our notation here.

Chapter 3 is devoted to the sporadic simple groups and their covering groups. Let G denote such a group. We determine all imprimitive irreducible KG-modules for fields K of arbitrary characteristic (Theorem 3.2). Thus this chapter contains the most complete results.

Next let G be a quasisimple group such that $G/Z(G)$ is an alternating group. There is a complete classification of the imprimitive absolutely irreducible KG-modules for fields K of characteristic 0. We cite the corresponding results from the literature in Chapter 4. The results for the alternating groups are due to D. Ž. Djoković and J. Malzan [**29, 30**] and are about 35 years old. More recent are the results for the covering groups due to D. Nett and F. Noeske [**90**]. Here, the classification over fields of positive characteristic remains open, although [**90**] restricts the possible block stabilizers.

The remaining chapters deal with the quasisimple covering groups of the finite groups of Lie type. Thus let G be a quasisimple group with $G/Z(G)$ a finite simple group of Lie type. The classification of the imprimitive irreducible KG-modules in the defining characteristic case, i.e., where the characteristic of K equals that of $G/Z(G)$, is due to Gary Seitz. There are only finitely many such instances. We state Seitz's result for completeness.

THEOREM 1.2. [**97**, Theorem 2] *Suppose that G is defined over a field of characteristic p. If M is an irreducible and imprimitive KG-module and* $\mathrm{char}(K) = p$, *then G is* $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{SL}_3(2)$ *or* $\mathrm{Sp}_4(3)$ *and M is the Steinberg module for G.*

There is one further case not covered in the above theorem, since it involves a group which has two defining characteristics, and is not

a simple gorup of Lie type in the strict sense. Namely, let $G = {}^2G_2(3)' \cong \mathrm{SL}_2(8)$, and $\mathrm{char}(K) = 3$. Then the three 9-dimensional irreducible KG-modules are imprimitive. In turn, the Steinberg module of ${}^2G_2(3)$ over K is induced from each of two 3-dimensional KH-modules, where H is the normalizer of Sylow 2-subgroup of ${}^2G_2(3)$.

From now on we will assume that the defining characteristic of $G/Z(G)$ is different from the characteristic of K. In all but finitely many cases, G is itself a finite group of Lie type. The exceptions occur if $G/Z(G)$ has an exceptional Schur multiplier. These cases are treated in Chapter 5, very much in the same way as the sporadic groups. Again, we obtain a complete classification for all characteristics with the only exception if $G/Z(G) = {}^2E_6(2)$ and $H/Z(G)$ is a parabolic subgroup of $G/Z(G)$, in which case we obtain this classification only for $\mathrm{char}(K) = 0$.

We are thus left with the cases that G and $G/Z(G)$ are finite groups of Lie type and that the characteristic of K is different from that underlying G. Our first main result here is Theorem 6.1: *If H is a maximal subgroup of G and a block stabilizer of an imprimitive irreducible KG-module, then H is a parabolic subgroup of G.* All of Chapter 6 is devoted to the proof of this theorem.

The remainder of our monograph is then devoted to the determination of imprimitive irreducible KG-modules whose block stabilizer is a parabolic subgroup H of G. We first prove in this situation that if M is induced from a KH-module M_1, then the unipotent radical of H acts trivially in M_1, in other words, M is Harish-Chandra induced (Proposition 7.1). Thus one can apply the well developed machinery of Harish-Chandra theory. This is exploited in Chapter 7 to give sufficient conditions on irreducible KH-modules to induce to irreducible KG-modules. These conditions are given in terms of the Harish-Chandra parametrization (Theorem 7.2) and of the Lusztig parametrization (Theorem 7.4) of the relevant irreducible KH-modules. These results are powerful enough to show that, asymptotically, most of the irreducible KG-modules are imprimitive (Theorem 7.6).

Let us comment on a consequence of this result on the problem of classifying maximal subgroups of a classical group X. Suppose that L is a subgroup of X of class $\mathcal{S}$ with $N_X(L) = L$ and $G := F^*(L)$ of Lie type of characteristic different from the characteristic of X. Suppose that the natural module V of X is an imprimitive Harish-Chandra induced KG-module with block stabilizer P. If the G-conjugacy class of P is L-invariant, the Frattini argument implies that $L = N_L(P)G$. Consequently, V is also an imprimitive KL-module. Indeed, let U denote the unipotent radical of P. As V is irreducible and Harish-Chandra

induced from P, there is a decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ such that $V_1 = C_V(U)$, and the V_i are permuted by the action of G. As U is fixed by $N_L(P)$, the KP-module V_1 is stabilized by $N_L(P)$. It now follows from $L = N_L(P)G$, that L permutes the V_i, and hence V is induced from the $KN_L(P)$-module V_1. Thus L is a $\mathcal{C}_2$-type subgroup of X, and as such, not maximal in X. Indeed, if $\tilde{L}$ is a maximal subgroup of X of $\mathcal{C}_2$-type such that $F^*(\tilde{L})$ is quasisimple, then $F^*(\tilde{L})$ does not act irreducibly on V. Therefore our results imply the surprising fact that most cross-characteristic irreducible modules for groups of Lie type do not in fact lead to examples of maximal subgroups of classical groups.

Harish-Chandra theory involves the investigation of representations of certain Iwahori-Hecke algebras. In case $\mathrm{char}(K) = 0$, this can be reduced to the investigation of representations of Weyl groups. This programme is carried out in Chapter 8, where we prove partial converses to the results of Chapter 7. In fact we show that the sufficient conditions derived there are necessary, if the groups G considered (Theorems 8.3 and 8.4) arise from algebraic groups with connected center. As a consequence, these results can not be applied to all quasisimple groups of Lie type.

The general results of Chapter 8 are now specialized to the classical groups in Chapter 9. Again we assume that $\mathrm{char}(K) = 0$, and that the groups G arise from algebraic groups with connected center. We obtain a complete classification of the Harish-Chandra imprimitive irreducible KG-modules in terms of Harish-Chandra series (Propositions 9.1 and 9.2) as well as in terms of Lusztig series (Propositions 9.4 and 9.5).

The final Chapter 10 is devoted to the exceptional groups of Lie type. We obtain a complete, characteristic independent and explicit classification of the imprimitive irreducible KG-modules if G is an exceptional group of Lie type of small rank, i.e., G is one of ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, $G_2(q)$, and ${}^3D_4(q)$. For the exceptional groups of Types E and F our results require the assumption $\mathrm{char}(K) = 0$.

CHAPTER 2

Generalities

In this chapter we start with some comments on our notation, in particular when it differs from that of our sources. Further notation is introduced as we go along. We also collect a few general results of various types, needed in later chapters. This chapter can be skipped at a first reading as it mainly serves as a convenient reference. Throughout, G denotes a group and K a field.

2.1. Comments on the notation

2.1.1. Notation for groups. Let G be a group. Our group actions are generally right actions. Thus for $x, y \in G$, we put

$$x^y := y^{-1}xy$$

and

$$[x, y] := x^{-1}y^{-1}xy = x^{-1}x^y.$$

If X and Y are subsets of G we write

$$[X, Y] := \langle [x, y] \mid x \in X, y \in Y \rangle.$$

Suppose that G is finite. If H a subgroup of G, then $[G\colon H]$ denotes the index of H in G. As usual, $F^*(G)$ denotes the generalized Fitting subgroup of G. Also, if ℓ is a prime, $O_\ell(G)$ is the largest normal ℓ-subgroup of G.

For group extensions we occasionally use the Atlas [**21**] convention, i.e., $A.B$ denotes a group G with a normal subgroup N isomorphic to A and G/N isomorphic to B. The symbol $A\colon B$ denotes a split extension, $A \times B$ the direct product of A and B, and $A \circ B$ a central product, i.e., a group G containing normal subgroups A and B with $[A, B] = 1$ and $AB = G$. A cyclic group of order n is sometimes just denoted by the symbol n.

The alternating and symmetric groups of degree n are denoted by A_n and S_n, respectively. The quaternion group of order 8 is denoted by Q_8. Our notation for the classical groups is more traditional than that of the Atlas. We follow essentially the notation used by Wilson in [**110**]. Thus $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{SL}_n(\mathbb{F})$, and $\mathrm{PSL}_n(\mathbb{F})$ denote, respectively, the general linear group, the special linear group and the projective special

linear group of degree n over the field $\mathbb{F}$. If $\mathbb{F}$ is finite with q elements, we write $\mathrm{GL}_n(q)$ etc. We write $\mathrm{GU}_n(q)$ for the unitary group which is a subgroup of $\mathrm{GL}_n(q^2)$. Our notation for orthogonal groups is as follows. If $n \geq 2$ is even, we write $\mathrm{GO}_n^+(q)$ and $\mathrm{GO}_n^-(q)$ for subgroup of $\mathrm{GL}_n(q)$ preserving a non-degenerate quadratic form on $\mathbb{F}_q^n$ of Witt index $n/2$ and $n/2-1$, respectively. If $n \geq 3$ is odd, $\mathrm{GO}_n(q)$ denotes the subgroup of $\mathrm{GL}_n(q)$ preserving a non-degenerate quadratic form on $\mathbb{F}_q^n$. If n is odd and q is even, by a non-degenerate quadratic form on $\mathbb{F}_q^n$ we mean a quadratic form whose polar form has a 1-dimensional radical consisting of non-isotropic vectors. For odd n we write $\mathrm{GO}_n^0(q) := \mathrm{GO}_n(q)$ so that $\mathrm{GO}_n^\epsilon(q)$ for $\epsilon \in \{0, +, -\}$ denotes our three types of orthogonal groups. Let $\epsilon \in \{0, +, -\}$. If q is odd, we put $\mathrm{SO}_n^\epsilon(q) := \mathrm{GO}_n^\epsilon(q) \cap \mathrm{SL}_n(q)$. We also put $\Omega^\epsilon(q) := [\mathrm{GO}_n^\epsilon(q), \mathrm{GO}_n^\epsilon(q)]$ for the commutator subgroup of $\mathrm{GO}_n^\epsilon(q)$. Finally, $\mathrm{Spin}_n^\epsilon(q)$ denotes the corresponding spin groups, i.e., the (generic) Schur covering groups of the $\Omega^\epsilon(q)$. Notice that $\Omega_n^\epsilon(q)$ is quasisimple if $n = 3$ and $q > 3$, if $n = 4$ and $\epsilon \neq +$, if $n = 5$ and $q > 2$, or if $n \geq 6$. For $n = 2$ we have that $\Omega_2^\epsilon(q)$ is cyclic of order $q-\epsilon$. If $\Omega_n^\epsilon(q)$ is quasisimple, the central factor group $P\Omega_n^\epsilon(q) := \Omega_n^\epsilon(q)/Z(\Omega_n^\epsilon(q))$ is simple. As $Z(\Omega_n^\epsilon(q))$ is trivial if q is even or n is odd, we just write $\Omega_n^\epsilon(q)$ for $P\Omega_n^\epsilon(q)$ in these cases.

Of course, there is also a version of these symbols emphasizing the underlying vector space carrying a form. For example, if V is a vector space equipped with a non-degenerate quadratic form, $\mathrm{GO}(V)$ denotes the orthogonal group of the quadratic space V, i.e., the automorphisms of V preserving the form $\mathrm{GO}(V)$.

2.1.2. Notation from linear algebra. If n is a positive integer, we denote by I_n the $n \times n$ identity matrix. We write J_n for the $n \times n$ matrix with ones along the anti-diagonal and zeros elsewhere. Finally, $\tilde{J}_{2n}$ denotes the $2n \times 2n$ matrix defined by

$$\tilde{J}_{2n} = \begin{pmatrix} 0_n & J_n \\ -J_n & 0_n \end{pmatrix},$$

where 0_n is the $n \times n$ zero matrix. These matrices are viewed as matrices over the ring currently considered. If n is clear from the context, we simply write I, J and $\tilde{J}$ for I_n, J_n and $\tilde{J}_{2n}$, respectively.

If A is a matrix, its transpose is denoted by A^T, and A^{-T} is defined by $A^{-T} := (A^{-1})^T$.

If a, b, c, ... are elements of a ring or square matrices over this ring (not necessarily of the same degree), we write $\mathrm{diag}(a, b, c, \ldots)$ for the (block) diagonal matrix with diagonal entries $a, b, c, \ldots$

Let V be a vector space and $G \leq \mathrm{GL}(V)$. If U is a G-invariant subspace of V and $x \in G$, we write x_U for the restriction of x to U, viewed as a linear map on U. Moreover, we write

$$G_U := \{x_U \mid x \in G\}.$$

2.1.3. Notation from representation theory. Let G be a finite group and K a field. By a KG-module we always mean a finite dimensional right KG-module. It will be convenient occasionally to consider the characters of KG-modules. If K is a splitting field for KG and $\mathrm{char}(K) = \ell$, we write $\mathrm{Irr}(G)$ for the set of characters of the irreducible KG-modules if $\ell = 0$, and $\mathrm{IBr}_\ell(G)$ for the set of Brauer characters (with respect to a fixed ℓ-modular system with residue class field K) of the irreducible KG-modules, if $\ell > 0$. If M and N are KG-modules, we put $[M, N]_{KG} = \dim_K \mathrm{Hom}_{KG}(M, N)$, omitting the subscript KG if there is no danger of confusion.

If H is a subgroup of G, we write $\mathrm{Ind}_H^G(-)$ and $\mathrm{Res}_H^G(-)$ for induction and restriction of KH-modules, respectively KG-modules. The same symbols are used for induction and restriction of characters.

2.2. Conditions for primitivity

Since we are interested in imprimitive irreducible KG-modules, we collect a few conditions guaranteeing that an induced KG-module is reducible. We begin with the most general one.

Lemma 2.1. *Let H be a subgroup of G. If $|H|^2 < |G|$, then $\mathrm{Ind}_H^G(M_1)$ is reducible for all KH-modules M_1.*

Proof. Let M_1 be a KH-module. Then

$$\dim_K(\mathrm{Ind}_H^G(M_1))^2 = \dim_K(M_1)^2\,[G:H]^2 > \dim_K(M_1)^2\,|G| \geq |G|.$$

As $\dim_K(N)^2 \leq |G|$ for all irreducible KG-modules N, our assertion follows. ◇

We shall need the following easy consequence of Mackey's theorem, which was also used implicitly in [**29, 30**].

Lemma 2.2. *Let H be a subgroup of G. Suppose that there exists an element $t \in G \setminus H$ such that t centralizes the intersection $H \cap H^t$. Then $\mathrm{Ind}_H^G(M_1)$ is reducible for all KH-modules M_1.*

Proof. By Mackey's theorem, the dimension of the endomorphism ring of $\mathrm{Ind}_H^G(M_1)$ is at least two. ◇

The following lemma gives a sufficient condition for the existence of an element t as above.

LEMMA 2.3. *Suppose that H and A are groups with $H \leq G \leq A$. Let $z \in A$ and put $C = C_G(z)$. Suppose that $s \in G$ such that $H \leq \langle C, s\rangle \leq N_A(C)$. Let $t \in G$ satisfy the following three conditions.*

(1) $t \in C_G(s)$;
(2) $t \in \langle z, z^t\rangle$;
(3) $C^t \cap Cs^i = \emptyset$ *for all* $i \in \mathbb{Z}$ *with* $s^i \notin C$.

Then $t \in C_G(H \cap H^t)$.

(The assumption $t \in \langle z, z^t\rangle$ is in particular satisfied if t has odd order and $t^z = t^{-1}$, since the latter implies $[z, t] = (t^z)^{-1}t = t^2$. Assumption (3) is trivially satisfied if $s = 1$.)

PROOF. As $t \in \langle z, z^t\rangle$ we have $t \in C_G(C \cap C^t)$. As t commutes with s we also have that t centralizes $\langle C \cap C^t, s\rangle$. We conclude by showing that $H \cap H^t \leq \langle C \cap C^t, s\rangle$.

We have $H \leq \langle C, s\rangle = C\langle s\rangle = \cup_{i\in\mathbb{Z}} Cs^i$. Similarly, $H^t \leq \cup_{i\in\mathbb{Z}} C^t s^i$. It follows that $H \cap H^t \leq \cup_{i,j\in\mathbb{Z}}(Cs^i \cap C^t s^j)$. By our assumption, $Cs^i \cap C^t s^j = \emptyset$, unless $s^{i-j} \in C$. Hence $H \cap H^t \leq \cup_{i\in\mathbb{Z}}(Cs^i \cap C^t s^i) = \langle C \cap C^t, s\rangle$. ◇

The case $s = 1$ is allowed in the above lemma. In fact this is the version of the lemma that we need to handle the $\mathcal{C}_1$-type subgroups. The case $s \neq 1$ is relevant when considering subgroups of types $\mathcal{C}_2$, $\mathcal{C}_3$ or $\mathcal{C}_5$.

Condition (3) of Lemma 2.3 is sometimes hard to verify. The following variation describes some favorable circumstances allowing to replace this verification.

LEMMA 2.4. *Assume the setup and the first two conditions of Lemma 2.3. Suppose also that s normalizes $\langle z\rangle$, and that $\langle t\rangle$ is a characteristic subgroup of $\langle z, z^t\rangle$. Then t centralizes $H \cap H^t$ if $N_G(\langle t\rangle) = C_G(t)$.*

PROOF. Since s normalizes $\langle z\rangle$, the latter group is also normalized by H. Hence H^t normalizes $\langle z^t\rangle$ and thus $H \cap H^t$ normalizes $\langle z, z^t\rangle$. It follows that $H \cap H^t \leq N_G(\langle t\rangle) = C_G(t)$. ◇

We will also need some generalizations of Lemma 2.3.

LEMMA 2.5. *Let S be a subgroup of G. Let $t \in G$ satisfy the following three conditions.*

(1) $t \in L := \langle S, S^t\rangle$.
(2) *There is a subgroup $D \leq G$ with $[L, D] = 1$, and there is $s \in G$ normalizing L and D such that $N_G(L) = (L \circ D).\langle s\rangle$.*
(3) *t centralizes $N_{\langle L,s\rangle}(S) \cap N_{\langle L,s\rangle}(S)^t$.*

Then t centralizes $N_G(S) \cap N_G(S)^t$.

Proof. Let $H := N_G(S)$ and $x \in H \cap H^t$. Then x normalizes S and S^t, hence L. As D centralizes L, it centralizes S and S^t and thus $D \leq H \cap H^t \leq N_G(L) = (L \circ D).\langle s \rangle$. Thus $H \cap H^t \leq \langle N_{\langle L,s\rangle}(S) \cap N_{\langle L,s\rangle}(S)^t, D\rangle$. Also, t centralizes D, as $t \in L$. The claim follows as t centralizes D and $(N_{\langle L,s\rangle}(S) \cap N_{\langle L,s\rangle}(S)^t)$. $\diamond$

Lemma 2.6. *Let Z be a subgroup of $Z(G)$ and $t \in G$ of order coprime to $|Z|$. Write $^{-}\colon G \to Z(G)$ for the canonical epimorphism. Then for all $x \in G$ such that $\bar{t} \in C_{\bar{G}}(\bar{x})$ we have $[x,t] = 1$.*

Proof. This follows from [**70**, 8.2.2.(b)]. $\diamond$

This lemma has the following simple, but important, corollary.

Corollary 2.7. *Let Z be a subgroup of $Z(G)$ and let $H \leq G$ with $Z \leq H$. Suppose that $t \in G$ is such that $\bar{t} \in \bar{G} = G/Z$ has order coprime to $|Z|$. Then $t \in C_G(H \cap H^t)$ if $\bar{t} \in C_{\bar{G}}(\bar{H} \cap \bar{H}^{\bar{t}})$.*

Proof. As $\bar{t}$ has order coprime to $|Z|$, there is an element $t_1 \in G$ with $\bar{t}_1 = \bar{t}$ and $|t_1| = |\bar{t}|$. As $t_1 = tz$ for some $z \in Z$, the claim follows from Lemma 2.6. $\diamond$

2.3. Some results on linear groups of small degree

We collect a couple of results, needed later on, on linear groups of degrees two and three.

Lemma 2.8. *Suppose that q is an odd prime power.*

(a) *Let $G \in \{\mathrm{GL}_2(q), \mathrm{GU}_2(q)\}$ and let $s \in G$ be a non-central involution. Then there are maximal tori T_1 and T_2 of $\{A \in G \mid \det(A) = 1\} \cong \mathrm{SL}_2(q)$ with $|T_1| = q-1$ and $|T_2| = q+1$ such that s inverts T_1 and T_2.*

(b) *Let $G = \mathrm{PGL}_2(q)$ and let $s \in G$ be an involution. Then there are maximal tori T_1 and T_2 of $\mathrm{PSL}_2(q)$ with $|T_1| = (q-1)/2$ and $|T_2| = (q+1)/2$ such that s inverts T_1 and T_2.*

Proof. (a) Suppose first that $G = \mathrm{GL}_2(q)$. As s is conjugate in G to the element
$$s_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
we may and will assume that $s = s_0$. Then
$$T_1 := \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{F}_q^* \right\}$$
is a torus of $\mathrm{SL}_2(q)$ of order $q-1$, inverted by s.

Next, let $x \in \mathbb{F}_{q^2}^*$ be an element of order $q+1$. The minimal polynomial of x over $\mathbb{F}_q$ has constant term equal to 1, as x and $x^q = x^{-1}$ are its zeroes. Define $a \in \mathbb{F}_q$ by $x^2 - ax + 1 = 0$. Then the matrix

$$z_0 := \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \tag{2.1}$$

represents right multiplication with x on $\mathbb{F}_{q^2}$ with respect to the basis $1, x$ of $\mathbb{F}_{q^2}$. Thus $T_2 := \langle z_0 \rangle$ is a torus of $\mathrm{SL}_2(q)$ of order $q+1$, which is obviously inverted by s.

Now suppose that $G = \mathrm{GU}_2(q) \leq \mathrm{GL}_2(q^2)$. We let the form defining G have Gram matrix I_2. Then $s_0 \in G$. Moreover, every involution of $G \setminus \mathrm{SU}_2(q)$ is conjugate to s_0 in G (since any two elements of G are conjugate in G if and only if they are conjugate in $\mathrm{GL}_2(q^2)$).

The intersection of T_1 (defined with respect to $\mathrm{GL}_2(q^2)$) with $\mathrm{SU}_2(q)$ has order $q+1$ and is inverted by s_0. The general theory of twisting tori (see [**19**, Section 3.3]) shows that there is $g \in \mathrm{GL}_2(\bar{\mathbb{F}}_{q^2})$, where $\bar{\mathbb{F}}_{q^2}$ denotes an algebraic closure of $\mathbb{F}_{q^2}$, such T_1^g (T_1 defined with respect to $\mathrm{GL}_2(q)$) lies in G, as well as s_0^g. As s_0^g is conjugate to s_0 in G, the result follows.

(b) It is well known that $\mathrm{PGL}_2(q)$ has exactly two conjugacy classes of involutions. If $s \notin \mathrm{PSL}_2(q)$, the claim follows from (a). If $s \in \mathrm{PSL}_2(q)$, the assertion follows from the fact that $\mathrm{PSL}_2(q)$ has dihedral subgroups of orders $q-1$ and $q+1$ (see [**59**, Sätze II.8.3, 8.4]). $\diamond$

LEMMA 2.9. *Let $L = \mathrm{PGL}_2(q)$ or $L = \mathrm{SL}_2(q)$ with $q > 3$ and let S be a maximal torus of L. Then there is $t \in L$ with $L = \langle S, S^t \rangle$ such that t centralizes $N_L(S) \cap N_L(S)^t$. If q is odd, there is such a t of odd order. If also $L = \mathrm{PGL}_2(q)$, then $L' = \langle S \cap L', (S \cap L')^t \rangle$ with $L' := [L, L] \cong \mathrm{PSL}_2(q)$.*

PROOF. Suppose first that q is odd. To begin with, let $L = \mathrm{PGL}_2(q)$. Then S contains a unique involution z and $N_L(S) = C_L(z)$. By Lemma 2.8(b) there is an element $1 \neq t \in L$ of odd order coprime to q such that $z^{-1}tz = t^{-1}$, since one of $q-1$ or $q+1$ is not a 2-power. Then $\langle z, z^t \rangle$ is a dihedral group of order $2|t|$ containing t. It follows from Lemma 2.3 that t centralizes $C_L(z) \cap C_L(z)^t = N_L(S) \cap N_L(S)^t$.

Now write L' for the subgroup of L of index 2, i.e., $L' = [L, L] \cong \mathrm{PSL}_2(q)$. Then $t \in L'$, as t has odd order. Put $S' := S \cap L'$. Then $N_L(S') = N_L(S)$ since $S = C_L(S')$. Thus t centralizes $N_{L'}(S') \cap N_{L'}(S')^t$. As t does not commute with $z \in S$, it follows that $t \notin S$, and thus $t \notin N_{L'}(S')$. Dickson's list of subgroup of L' (see [**59**, II, Hauptsatz 8.27]) now implies that $\langle S', (S')^t \rangle = L'$. Since $S \not\leq L'$, we obtain $\langle S, S^t \rangle = L$, proving all our assertions for q odd and $L = \mathrm{PGL}_2(q)$.

Now let $L = \mathrm{SL}_2(q)$ (still assuming that q is odd), and write $^-: \mathrm{SL}_2(q) \to \mathrm{PSL}_2(q)$ for the canonical epimorphism. By what we have already proved, there is an element $\bar{t} \in \bar{L}$ of odd order such that $\bar{L} = \langle \bar{S}, \bar{S}^{\bar{t}} \rangle$ and such that $\bar{t}$ centralizes $N_{\bar{L}}(\bar{S}) \cap N_{\bar{L}}(\bar{S})^{\bar{t}}$. Let $t \in L$ be a preimage of $\bar{t}$ of odd order. As L is a non-split central extension of $\bar{L}$, we have $L = \langle S, S^t \rangle$. Moreover, t centralizes $N_L(S) \cap N_L(S)^t$ by Corollary 2.7.

Now suppose that q is even. Then $\mathrm{PGL}_2(q) \cong \mathrm{SL}_2(q)$ and $N_L(S)$ is a dihedral group of twice odd order. Let $z \in N_L(S)$ be an involution. Choose an involution $t \in L$ centralizing z and not equal to z. (This is possible since $q \neq 2$.) Then $t \notin N_L(S)$ since no two distinct involutions of $N_L(S)$ commute. As S is a TI subgroup of L, it follows that $N_L(S) \cap N_L(S)^t = \langle z \rangle$, and thus t commutes with this intersection.

In order to show that $\langle S, S^t \rangle = L$, we distinguish two cases. If S is a Coxeter torus of L, then $\langle S, S^t \rangle = L$ since any two disjoint Coxeter tori of L generate L. If S fixes a line in the natural representation of L on $\mathbb{F}_q^2$, i.e., L lies in some Borel subgroup B of L, then $\langle S, S^t \rangle = L$. Otherwise, $\langle S, S^t \rangle = B$. But z normalizes $\langle S, S^t \rangle = B$, hence $z \in B$. This is impossible as $N_B(S) = S$. $\diamond$

Lemma 2.10. *Let $G = \mathrm{GL}_3(q)$, and $S \leq G$ a natural $\mathrm{SL}_2(q)$-subgroup. Then there exists a $t \in L := \mathrm{SL}_3(q) \leq G$ such that t centralizes $N_G(S) \cap N_G(S)^t$ and such that $L = \langle S, S^t \rangle$.*

Proof. Put $N := N_G(S)$. Then N fixes a unique 2-space U of the natural module V of G and a unique 1-space $\langle v \rangle$ lying outside of U. Conversely, N is the stabilizer of the pair of subspaces U and $\langle v \rangle$.

Now let e_1, e_2, e_3 be a basis of V and put $U_1 := \langle e_1, e_2 \rangle$, $U_2 := \langle e_2, e_3 \rangle$, $v_1 = e_3$, $v_2 = e_1 + e_3$. Assume that N is the stabilizer of U_1 and $\langle v_1 \rangle$. If $t \in G$ is the linear transformation defined by $v_1 t = v_2$, $e_2 t = e_2$ and $e_1 t = e_3 = v_1$, then $U_1 t = U_2$ and $v_1 t = v_2$. Thus N^t is the stabilizer of U_2 and $\langle v_2 \rangle$. It follows that $N \cap N^t$ stabilizes $U_1 \cap U_2 = \langle e_2 \rangle$, $\langle v_1 \rangle$ and $\langle v_2 \rangle$. Let $x \in N \cap N^t$ and suppose that $v_1 x = a v_1$ and $v_2 x = b v_2$ for some $a, b \in \mathbb{F}_q$. Then $a = b$, since x stabilizes U_1. This implies that t commutes with x, as the vectors e_2, v_1, v_2 form a basis of V and $v_2 t = v_1 + v_2$.

Finally we observe that the $S = \mathrm{SL}(U_1)$ and the group generated by $\mathrm{SL}(U_1)$ and $\mathrm{SL}(U_2) = S^t$ contains a generating set of transvections for $L = \mathrm{SL}_3(q)$, and the lemma is proved. $\diamond$

2.4. Reduction modulo ℓ and imprimitivity

So far our preliminary results have been independent of the characteristic of K. We will also need the following elementary observations

relating imprimitivity of irreducible characters in different characteristics through reduction modulo ℓ. Here, we assume that K is large enough, i.e., that K is a splitting field for all subgroups of G.

Let ℓ be a prime and let χ be an ordinary irreducible character of G, i.e., $\chi \in \mathrm{Irr}(G)$. The restriction of χ to the ℓ-regular conjugacy classes of G is called the *reduction of* χ *modulo* ℓ. If K has characteristic ℓ, we call an irreducible KG-module *liftable*, if its Brauer character (with respect to a suitable ℓ-modular system) is obtained by reduction modulo ℓ of some ordinary irreducible character of G. We refer the reader to [**72**, Section I.14] for more general definitions and properties of reduction modulo ℓ and liftability of modules.

LEMMA 2.11. *Let H be a proper subgroup of G. Let $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G)$. Furthermore, let $\varphi \in \mathrm{IBr}_\ell(H)$ and $\vartheta \in \mathrm{IBr}_\ell(G)$. Then the following hold.*

(1) *Suppose that φ is the reduction modulo ℓ of ψ and that $\mathrm{Ind}_H^G(\psi)$ is reducible. Then $\mathrm{Ind}_H^G(\varphi)$ is also reducible.*

(2) *Suppose that φ lifts to ψ and that $\mathrm{Ind}_H^G(\varphi)$ is irreducible. Then $\mathrm{Ind}_H^G(\varphi)$ lifts to the imprimitive irreducible ordinary character $\mathrm{Ind}_H^G(\psi)$.*

(3) *Suppose that $\chi = \mathrm{Ind}_H^G(\psi)$, i.e., χ is imprimitive. If ϑ is the reduction modulo ℓ of χ, then ϑ is also imprimitive.*

(4) *Suppose that ψ is of ℓ-defect zero and that $\chi = \mathrm{Ind}_H^G(\psi)$. Then χ is of ℓ-defect zero and the reduction modulo ℓ of χ is imprimitive.*

PROOF. Parts (1) and (3) are clear since induction of characters commutes with reduction modulo ℓ, i.e., restriction to ℓ-regular elements. Part (2) is just a restatement of Part (1) from a different point of view. Finally, Part (4) is a special case of Part (3). $\diamond$

These observations allow us to eliminate many possible block stabilizers H and characters ψ. In particular, the linear characters of H remain irreducible in any characteristic (although they may become trivial). Moreover all modular linear characters arise as restrictions of some ordinary linear character. Thus, knowledge of the behavior of the ordinary linear characters of H is sufficient to determine their modular behavior.

We also need a generalization of Lemma 2.11.

LEMMA 2.12. *Let $\psi \in \mathrm{Irr}(H)$ and $\varphi \in \mathrm{IBr}_\ell(H)$ such that φ occurs in the reduction modulo ℓ of ψ. If $\chi(1) < [G\colon H]\varphi(1)$ for all irreducible constituents χ of $\mathrm{Ind}_H^G(\psi)$, then $\mathrm{Ind}_H^G(\varphi)$ is reducible.*

PROOF. Again we use the fact that induction commutes with reduction modulo ℓ. If $\mathrm{Ind}_H^G(\varphi)$ were irreducible, it would be a constituent of

the reduction modulo ℓ of some irreducible constituent χ of $\mathrm{Ind}_H^G(\psi)$. $\diamond$

2.5. A result on polynomials

Finally, we record a lemma on the evaluation of polynomials which will be used later on.

LEMMA 2.13. *Let* $f = \sum_{i=0}^{n} a_i X^i \in \mathbb{R}[X]$ *be a non-zero polynomial of degree* n *(i.e.,* $a_n \neq 0$*). Put* $B := \max\{|a_i|/|a_n| \mid 0 \leq i \leq n\}$ *and let* $b \in \mathbb{R}$ *with* $b \geq B + 1$*.*

Then $f(b) \neq 0$*. Moreover,* $f(b) < 0$ *if and only if* $a_n < 0$*.*

PROOF. We may assume that $|a_n| = 1$ and thus

$$B = \max\{|a_i| \mid 0 \leq i \leq n\}.$$

Then

$$\left|\sum_{i=0}^{n-1} a_i b^i\right| \leq \sum_{i=0}^{n-1} |a_i| b^i \leq B\frac{b^n - 1}{b - 1} \leq b^n - 1 < b^n.$$

This implies both results. $\diamond$

CHAPTER 3

Sporadic Groups

In this chapter G is a quasisimple group such that $G/Z(G)$ is a sporadic simple group, and H is a maximal subgroup of G. As always, K denotes an algebraically closed field of characteristic $\ell \geq 0$. Our arguments here are based on characters rather than modules or representations, so we work with the irreducible K-characters of H and G throughout. We determine all imprimitive irreducible K-characters of G and the corresponding maximal block stabilizers. Of course, it suffices to consider the cases $\ell = 0$ and $\ell \mid |G|$. We mostly apply ad hoc methods, although we use the same general approach described below.

Step 1: Using a list of maximal subgroups of G and the ordinary character tables of G as found in the Atlas [**21**] or GAP [**37**], we generate a list of all maximal subgroups whose index is smaller than the maximal degree of an ordinary irreducible character for G. Note that every possible maximal block stabilizer, in any characteristic, must occur on this list. Moreover we produce an upper bound on the degree of a K-character of H which may induce to an irreducible K-character of G.

Step 2: If the degrees of the modular irreducible characters of G are known, then we refine the list generated in Step 1 by checking whether the index of a given maximal subgroup divides the degree of some irreducible character of G. We also determine the exact degree of a possible imprimitive character.

Step 3: If the character table for the maximal subgroup H is also known, we induce the characters of H to G using GAP and determine if any of the induced characters are irreducible.

Step 4: The remaining groups on our list that can not be approached using Step 3 must be dealt with separately.

More details are given in the proof of Theorem 3.2 below, which is the main result of this chapter. The instances of imprimitive irreducible characters are presented in Table 1. We are now going to describe how to read this table. More explanations on particular entries are given in Remark 3.1 below.

The notation in the table follows, as far as possible, the Atlas [**21**], except that we have chosen a more traditional notion for the classical groups. The first two columns describe G and H, respectively. The third column, headed ψ, indicates the irreducible character of H which induces to the irreducible character χ of G, identified in the sixth column. In those cases where H is given as a direct product $H_1 \times H_2$, ψ is written as an outer product of the form $\psi_1 \boxtimes \psi_2$, with ψ_i an irreducible character of H_i, $i = 1, 2$. Whenever H or H_i is an Atlas table, we use the Atlas notation of the compound character table to describe ψ or ψ_i, respectively. Similarly, the character χ of G is identified. With ζ_i we denote a linear K-character of order i, $i = 2, 3, \ldots$ Some other characters are denoted by their degrees, usually with subscripts to distinguish characters of the same degrees. There should be no problem to match the characters with existing tables. The degrees of ψ and χ are given in the columns labelled by $\psi(1)$ and $\chi(1)$, respectively.

Given G, there is exactly one block of rows for the *faithful* irreducible imprimitive characters of G. These blocks are separated by two horizontal rules. As in the Atlas, a single row can describe more than one irreducible character of H or G. This can be detected from the "ind"-columns, giving the Frobenius-Schur indicators and, possibly, the number of the characters in a row. The characteristic is always odd in Table 1, so that these indicators can easily be determined.

The column headed "Primes" lists those characteristics, for which the example described in the respective row exists for one of the reasons described in Lemma 2.11.

Remark 3.1. M_{11}: The character 1_3 of the maximal subgroup $H = 3^2{:}Q_8.2$ is the non-trivial linear character which has the elements of order 8 of H in its kernel.

M_{12} and $2.M_{12}$: The character 1_3 of the maximal subgroup $H = M_{10}{:}2$ of M_{12} is the non-trivial linear character which has the elements of order 10 of H in its kernel.

There are two conjugacy classes of subgroups of M_{12} isomorphic to M_{11} and two conjugacy classes of subgroups isomorphic to $M_{10}{:}2$. In each case, the two conjugacy classes are swapped by the outer automorphism. Since the irreducible imprimitive characters of G induced from these subgroups are invariant under the outer automorphism, these characters are also induced from the conjugate subgroups.

M_{22}: Note that the example in characteristic 3 arises from the reduction modulo 3 of the character χ_{40} of $3.M_{22}$ (Atlas notation). Since this reduction is not faithful anymore, we have introduced it in an extra row of Table 1.

$3.M_{22}$: The characters χ_{13} and χ_{14} are the inflations to $3.(2^4{:}A_6)$ of the characters of degree 3 of $3.A_6$.

Co_2: The character ψ_3' is (the inflation to $2^{10}{:}(M_{22}{:}2)$ of) the extension to $M_{22}{:}2$ of ψ_3 with the property that its value on class $2B$ equals -3.

THEOREM 3.2. (1) *The following groups have no irreducible and imprimitive characters in any characteristic:* J_1, $2.J_2$, M_{23}, $3.J_3$, He, 2.Ru, 6.Suz, Co_3, HN, Ly, Th, Fi_{23}, J_4, $2.B$, *and* M.

(2) *All examples of irreducible imprimitive characters for the groups* M_{11}, $2.M_{12}$, $12.M_{22}$, 2.HS, M_{24}, 3.McL, 3.O'N, Co_2, $6.\mathrm{Fi}_{22}$, $2.\mathrm{Co}_1$ *and* $3.\mathrm{Fi}_{24}'$ *are known and are listed in Table 1.*

PROOF. Our proof proceeds by completing Steps 1–4 above for all of the sporadic simple groups and their covering groups. In this proof, by an *example* we always mean an imprimitive irreducible character of G. Characters can be ordinary or modular characters, and ℓ denotes the characteristic of the underlying field K, where $\ell = 0$ is allowed. If H is a maximal subgroup of H, a (hypothetical) character of H inducing to an irreducible character of G is denoted by φ, if ℓ is not specified, and by ψ, if $\ell = 0$.

$G \in \{M_{11}, 2.M_{12}, 2.J_2, M_{23}, 2.\mathrm{HS}, 3.\mathrm{McL}, \mathrm{He}\}$: All the character tables are known in GAP for both G and for all maximal subgroups on the list generated in Step 2. Here all examples can be determined using GAP.

$G = J_1$ or $G = 3.J_3$: The list of maximal subgroups for G generated in Step 2 is empty, hence no examples exist.

$G = 12.M_{22}$: The modular character tables of $12.M_{22}$ modulo all primes are known as well as the ordinary character tables of all maximal subgroups of $12.M_{22}$. It is easy to check using GAP and Lemma 2.11, that the examples of imprimitive irreducible characters given in Table 1 for the various central quotient groups of $12.M_{22}$ are correct.

There are a few candidates for imprimitive irreducible characters of G which do not yield examples. However, the potential characters which could possibly induce to irreducible characters of G have small degrees, and these hypothetical cases are easily ruled out by ad hoc arguments. We only give one such argument here. The group $G = 2.M_{22}$ has two faithful irreducible characters of degree 154 in characteristic $\ell = 3$. Let ϑ be one of them, and let $H = 2.(2^4 : A_6)$ be the maximal subgroup of G of index 77. Suppose that φ is an irreducible character of H of degree 2, inducing to ϑ. Then $Z(G)$ is not in the kernel of φ. All ordinary irreducible characters of H with $Z(G)$ not in their kernels

TABLE 1. Sporadic group examples

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes
M_{11}	M_{10}	ζ_2	$+$	1	χ_5	$+$	11	0, 5, 11
	$3^2\!:\!Q_8.2$	1_3	$+$	1	χ_{10}	$+$	55	0, 5, 11
M_{12}	M_{11}	ψ_3	$\circ 2$	10	χ_{13}	$+$	120	0, 5
	$M_{10}\!:\!2$	1_3	$+$	1	χ_{11}	$+$	66	0, 5, 11
$2.M_{12}$	$2 \times M_{11}$	$\zeta_2 \boxtimes \psi_1$	$+$	1	χ_{18}	$+$	12	0, 5, 11
		$\zeta_2 \boxtimes \psi_2$	$+$	10	χ_{24}	$+$	120	0, 5
		$\zeta_2 \boxtimes 9_1$	$+$	9	108_1	$+$	108	11
M_{22}	$2^4\!:\!A_6$	3_1	$+$	3	231_1	$+$	231	3
		3_2	$+$	3	231_1	$+$	231	3
$3.M_{22}$	$3.\mathrm{PSL}_3(4)$	ψ_{33}	$\circ 2$	15	χ_{41}	$\circ 2$	330	0, 5, 7, 11
		ψ_{34}	$\circ 2$	15	χ_{41}	$\circ 2$	330	0, 5, 7, 11
	$3.(2^4\!:\!A_6)$	ψ_{14}	$\circ 2$	3	χ_{40}	$\circ 2$	231	0, 5, 7, 11
		ψ_{15}	$\circ 2$	3	χ_{40}	$\circ 2$	231	0, 5, 7, 11
	$3 \times (2^4\!:\!S_5)$	ζ_6	$\circ 2$	1	χ_{39}	$\circ 2$	231	0, 5, 7, 11
$4.M_{22}$	$4_1.\mathrm{PSL}_3(4)$	ψ_{19}	$\circ 2$	8	χ_{30}	$\circ 2$	176	0, 11
		ψ_{20}	$\circ 2$	8	χ_{30}	$\circ 2$	176	0, 11
$6.M_{22}$	$6 \times (2^3\!:\!\mathrm{SL}_3(2))$	ζ_6	$\circ 2$	1	χ_{51}	$\circ 2$	330	0, 5, 7, 11
2.HS	$2.M_{22}$	ψ_{13}	$\circ 2$	10	χ_{32}	$-$	1 000	0, 3, 5
	$\mathrm{PSU}_3(5).(2 \times 2)^{\imath}$	ζ_4	$\circ 2$	1	χ_{26}	$\circ 2$	176	0, 3, 7, 11
M_{24}	$2^6\!:\!3.S_6$	ζ_2	$+$	1	χ_{18}	$+$	1 771	0, 5, 7, 11, 23
McL	$\mathrm{PSU}_4(3)$	ψ_3	$+$	35	χ_{20}	$+$	9 625	0, 5, 7, 11
		ψ_4	$+$	35	χ_{20}	$+$	9 625	0, 5, 7, 11
3.O'N	$3 \times (\mathrm{PSL}_3(7)\!:\!2)$	$\zeta_3 \boxtimes \zeta_2$	$\circ 2$	1	χ_{45}	$\circ 2$	122 760	0, 5, 11, 31
Co_2	$\mathrm{PSU}_6(2).2$	ψ_7	$\circ 2$	560	χ_{50}	$+$	1 288 000	0, 5, 7, 23
	$2^{10}\!:\!(M_{22}\!:\!2)$	ψ_3'	$\circ 2$	45	χ_{60}	$+$	2 095 875	0, 3, 5, 23
$6.\mathrm{Fi}_{22}$	$6 \times \Omega_8^+(2)\!:\!S_3$	$\zeta_6 \boxtimes 1$	$\circ 2$	1	χ_{167}	$\circ 2$	61 776	0, 5, 11, 13
		$\zeta_6 \boxtimes \zeta_2$	$\circ 2$	1	χ_{168}	$\circ 2$	61 776	0, 5, 11, 13
		$\zeta_6 \boxtimes 2$	$\circ 2$	2	χ_{173}	$\circ 2$	123 552	0, 5, 11, 13
$2.\mathrm{Co}_1$	$2 \times \mathrm{Co}_2$	$\zeta_2 \boxtimes \psi_{12}$	$\circ 2$	10 395	χ_{167}	$+$	1 021 620 600	0, 7, 11, 13
$3.\mathrm{Fi}_{24}'$	$3 \times \mathrm{Fi}_{23}$	$\zeta_3 \boxtimes \psi_{15}$	$\circ 2$	837 200	χ_{175}	$\circ 2$	256 966 819 200	0, 5, 7, 13, 23, 29
		$\zeta_3 \boxtimes \psi_{16}$	$\circ 2$	837 200	χ_{175}	$\circ 2$	256 966 819 200	0, 5, 7, 13, 23, 29

are faithful (see GAP). Now φ is a $\mathbb{Z}$-linear combination of such ordinary irreducible characters (restricted to the 3-regular classes of H). Since H does not have any non-trivial normal 3-subgroup, this implies that φ is faithful. Since the character table of H has only quadratic irrationalities, this would yield an embedding of H into $\mathrm{GL}_2(9)$, which is absurd.

$G = M_{24}$: The ordinary character tables of the maximal subgroups of M_{24} are available in GAP, as well as enough of the modular character tables, so that the result can be derived with the help of Lemma 2.11.

$G = 2.\mathrm{Ru}$: The only maximal subgroups that could lead to examples are ${}^2F_4(2)'$ and $2^6 : \mathrm{PSU}_3(3) : 2$. Moreover, all degrees of irreducible 2.Ru characters are known. We find that the only possibilities are for a 13 or 26-degree character of ${}^2F_4(2)'$ in characteristic 0 and 7. As 7 is coprime to the order of ${}^2F_4(2)'$, we only need to check the ordinary characters using GAP. Here we find no examples.

$G = 6.\mathrm{Suz}$: The available character tables in GAP and Lemma 2.11 leave only one possibility for a potential imprimitive irreducible character of G, a faithful character ϑ of degree 68 640 in characteristic $\ell = 5$, with block stabilizer H of index 22 880. Suppose that φ is an irreducible character of H inducing to ϑ. Then $\varphi(1) = 3$. Since ϑ is faithful, $Z(G)$ is faithfully represented on a module with character φ. But $Z(G) = Z(H) \leq H'$ (see GAP) implies that the degree of φ is divisible by 6, a contradiction.

$G = 3.\mathrm{O'N}$: The only subgroup with an index which is small enough is $3 \times (\mathrm{PSL}_3(7) : 2)$. Moreover, the only character that may work is linear. Checking the ordinary characters, we see that exactly the two linear characters of $3 \times (\mathrm{PSL}_3(7) : 2)$ of order 6 induce to irreducible characters of 3.O'N. The corresponding ordinary irreducible imprimitive characters of 3.O'N have degree 122 760. Since these two characters are reducible modulo ℓ for $\ell \in \{2, 3, 7, 19\}$, and irreducible modulo all other primes, the results of Table 1 follow from Lemma 2.11.

$G = \mathrm{Co}_3$: The modular character tables for G are known, as well as the ordinary character tables of all maximal subgroups of G. Using this information, we find that only the largest maximal subgroup McL.2 can possibly be a block stabilizer. But all modular character tables for McL.2 are known as well, so that this case is easily ruled out.

$G = \mathrm{Co}_2$: The four possible maximal block stabilizers are $\mathrm{PSU}_6(2).2$, $2^{10} : M_{22} : 2$, McL and $2^{1+8} : \mathrm{Sp}_6(2)$. The modular character tables of $\mathrm{PSU}_6(2).2$ are available in GAP, which makes it easy to complete the proof in this case. For $H \in \{2^{10} : M_{22} : 2, \mathrm{McL}, 2^{1+8} : \mathrm{Sp}_6(2)\}$, we find that the only characters in any characteristic that could induce to irreducible characters are irreducible restrictions of characteristic 0 examples. Using Lemma 2.11 and GAP, we find the examples in these cases.

$G = 6.\mathrm{Fi}_{22}$: All modular character tables for G are known (see [**51, 91**]). Only the five largest maximal subgroups remain after Step 1, and the smallest of these, M_5 say, of index 142 155, does not give any

candidates inducing to faithful characters of G. In fact the only candidates for M_5 have degree 6, and this possibility only occurs for Fi_{22} and $3.\mathrm{Fi}_{22}$ and in characteristics $\ell \neq 2$. Now M_5 has a normal subgroup N isomorphic to 2^{10}, and any ℓ-modular irreducible character of M_5 with N not in its kernel has degree larger than 6 by Clifford theory. Hence the candidates for M_5 have N in their kernel and are thus irreducible ℓ-modular characters of M_{22} or $3.M_{22}$, respectively. But these groups do not have irreducible ℓ-modular characters of degree 6.

The ordinary character tables of the four largest maximal subgroups are also available in GAP, as well as their modular characters. This allows us easily to deduce the result.

$G = \mathrm{HN}$: Here, the only subgroups with an index smaller than the largest irreducible character of G are A_{12} and 2.HS.2. Moreover, the upper bound for the degree is 5 for an A_{12}-character and 3 for a 2.HS.2-character. Thus, an example can arise only from a linear character of 2.HS.2. Checking the characteristic 0 case and using Lemma 2.11, we find that there are no examples for this group.

$G = \mathrm{Ly}$: The only possibilities for H are $G_2(5)$ and 3.McL.2. Moreover, the upper bound for the degree of an H-character is 8 in both cases. If $H = 3.\mathrm{McL}.2$, then only linear characters are possible. Checking the characteristic 0 case and using Lemma 2.11, we see that H is not a block stabilizer.

If $H = G_2(5)$, then a character φ inducing irreducibly to G must be the character of the minimal module for $G_2(5)$ in characteristic 5. Now there is a conjugate H^t of H such that $H^t \cap H = \mathrm{PSU}_3(3)$ (see [**21**, p. 174]). Moreover the restriction to this intersection of the minimal module for H is irreducible. Although $\mathrm{PSU}_3(3)$ has three 5-modular characters of degree 7, the restriction of the minimal character of H has value 0 on all 12-elements of $\mathrm{PSU}_3(3)$, but no other 5-modular character of degree 7 of $\mathrm{PSU}_3(3)$ has this property (irreducible or reducible). In particular, $\mathrm{Res}_{H^t \cap H}(\varphi) = \mathrm{Res}_{H^t \cap H}({}^t\varphi)$, which eliminates this case as well.

$G = \mathrm{Th}$: The only subgroup that may provide an example is ${}^3D_4(2){:}3$ with a linear character. Using the ordinary character tables and Lemma 2.11, we see that there are no examples for this group.

$G = \mathrm{Fi}_{23}$: According to the "Improvements to the Atlas" in [**65**, Appendix 2], the list of maximal subgroups of Fi_{23} given in [**21**, p. 177] is complete, except that the last maximal subgroup has to be replaced by $\mathrm{PSL}_2(23)$ (see [**65**, p. 304]). A potential block stabilizer is among the first six maximal subgroups. We consider each of these possibilities in turn, starting with the smallest one. In all cases, we denote by ψ an irreducible K-character inducing irreducibly to G.

Suppose first that $H = 2^{11}.M_{23}$. Then $\psi(1) \leq 2$ and, since H is perfect, we have $\psi(1) = 2$. Since H does not have any irreducible character of this degree, this subgroup does not provide any examples.

Next let $H = S_3 \times \Omega_7(3)$. Here, $\psi(1) \leq 3$. Since the smallest degree of a non-trivial irreducible character of $\Omega_7(3)$ is 7, we must have $\Omega_7(3)$ in the kernel of ψ, i.e., ψ is a character of S_3. In S_3, every modular irreducible character lifts to an ordinary irreducible character. By the first part of Lemma 2.11, it suffices to check the ordinary character tables. This does not give any example.

If $H = \mathrm{Sp}_8(2)$, we have $\psi(1) \leq 7$, but a non-trivial irreducible character of H has degree at least 8. Thus we do not find an example with block stabilizer $\mathrm{Sp}_8(2)$.

Now let $H = 2^2.\mathrm{PSU}_6(2).2$. In this case $\psi(1) \leq 10$. To show that only characters of degree 1 satisfy this condition, it suffices to look at the subgroup $2^2.\mathrm{PSU}_6(2)$ of H. This group does not have any faithful irreducible characters, and a non-trivial irreducible character of $2.\mathrm{PSU}_6(2)$ has degree at least 20. Hence $\psi(1) = 1$. Checking the ordinary character table and using Lemma 2.11, we find no example for this subgroup either.

Next let $H = P\Omega_8(3)^+ : S_3$. Put $F := \mathrm{Fi}_{24}$ and let u, v, w be 3-transpositions of F such that $uv = vu$, but w does not commute with u and v. Then $G = C_{F'}(u)$, and $H = C_G(uw) = C_{F'}(u, w)$ (see [**4**, (16.12), (15.14), (25.9)]). Put $L := C_{F'}(u, v, w)$. Then $L \cong \Omega_8^+(2) : S_3$ by [**4**, (25.6)]. Put $s := uv$. Then $s \in \langle u, v, w\rangle$ and thus s centralizes L. It follows that $L = H \cap H^s$ since L is a maximal subgroup of H (see [**21**, p. 140]). Hence H is not a block stabilizer by Lemma 2.2.

Finally, let $H = 2.\mathrm{Fi}_{22}$, the largest maximal subgroup of G. Let $t \in G$ be an element of the $3A$-conjugacy class of G. Then $N := N_G(\langle t\rangle) = \langle t, s\rangle \times L$ with $\langle t, s\rangle \cong S_3$ and $L \cong \Omega_7(3)$ (see [**21**, p. 177]). Thus s is an involution in class $2A$, since no other involution centralizer is divisible by $|L|$. We choose $H = C_G(s)$. Then $L \leq H^t \cap H$. Since $\langle s\rangle \times L$ is a maximal subgroup of H (see [**21**, p. 163]), and since $s^t \notin H$, we have $L = H^t \cap H$. In particular, t centralizes $H^t \cap H$. Hence H is not a block stabilizer by Lemma 2.2.

$G = 2.\mathrm{Co1}$ or $G = \mathrm{Co}_1$: Suppose first that $\ell = 0$. Step 1 leaves the seven largest maximal subgroups as candidates for block stabilizers, but only the largest and the fifth largest maximal subgroup remain after Step 2. For the latter subgroup the only candidates have degree 1. Now the subgroup $2_+^{1+8}.\Omega_8^+(2)$ of Co_1 is perfect and thus cannot be a block stabilizer. Moreover, $2_+^{1+8}.\Omega_8^+(2)$ is not a subgroup of $2.\mathrm{Co}_1$, as

can be shown with GAP. Hence $2.(2_+^{1+8}.\Omega_8^+(2)) \leq 2.\mathrm{Co}_1$ is perfect as well, and so it cannot be a block stabilizer either.

Next let $H = 2 \times \mathrm{Co}_2$ be the largest maximal subgroup of $2.\mathrm{Co}_1$. The only candidates have degrees 1 080, 5 313 and 10 395, and since 1 080 and 5 313 are not character degrees for H, only 10 395 remains. These candidates yield the characteristic 0 examples of Table 1. The modular character tables of H and of $2.\mathrm{Co}_1$ are known for $\ell = 7, 11, 13$ and 23 (see [**50**] and [**13**]). This allows us to conclude that the degree 10 395 characters also yield examples for $\ell = 7, 11$ and 13 and that there are no more examples with block stabilizer H in these characteristics.

We are now ging to show that there are no more examples of irreducible imprimitive characters. Assume from now on that $\ell \neq 0$. We consider the seven maximal subgroups of $2.\mathrm{Co}_1$ in turn, starting with the smallest one.

This is $H := 2.(A_4 \times G_2(4)){:}\,2)$. Here, $\varphi(1) = 1$ and thus φ is liftable. But $2.\mathrm{Co}_1$ does not have an ordinary irreducible character of degree equal to the index of H. Thus H does not give any example.

Next, let $H = 2.(\mathrm{PSU}_6(2){:}\,S_3)$. Here, $\varphi(1) \leq 13$. Since the non-split extension $2.\mathrm{PSU}_6(2)$ is not a subgroup of $2.\mathrm{Co}_1$ (use GAP to show this), it follows that H has a normal subgroup L of index 12, isomorphic to $\mathrm{PSU}_6(2)$. The smallest non-trivial character degree of $\mathrm{PSU}_6(2)$ in any characteristic is larger than 13 (see [**13**]). Hence φ has L in its kernel and is thus liftable. Moreover, $\varphi(1) \in \{1, 2, 3\}$. But $2.\mathrm{Co}_1$ does not have an ordinary irreducible character of degree equal to $d \cdot [2.\mathrm{Co}_1 : H]$ for $d = 1, 2, 3$, and so H does not give any example.

Now let $G = \mathrm{Co}_1$ and $H = 2_+^{1+8}.\Omega_8^+(2)$. Then $H = C_G(x)$ for an element z in class $2A$ of G. By computing class multiplication coefficients we see that there is a conjugate y of z such that zy lies in class $3B$. Hence there is an element t of order 3 such that $t \in \langle z, z^t \rangle$. Lemma 2.3 and Corollary 2.7 impliy that neither H nor its inverse image in $2.G$ are block stabilizers of imprimitive irreducible representations of G, respectively $2.G$.

Next let $G = 2.\mathrm{Co}_1$ and $H = 2 \times \mathrm{Co}_3$. All modular character tables of H are known. From these we conclude that the only candidates φ, which are not liftable have degree 22, and occur for $\ell = 2$ and 3. In this case, φ is a constituent of the reduction modulo ℓ of one of the two ordinary characters of H of degree 23. We can use Lemma 2.12 to show that H is not a block stabilizer.

Now let $G = 2.\mathrm{Co}_1$ and $H = 2.(2^{11} : M_{24})$. Here, $\varphi(1) \leq 123$. Using Clifford theory and the known modular character tables of M_{24}, we find that the only non-liftable Brauer characters φ of degrees at most 123

have degrees 22 if $\ell = 3$, and 11 and 44 if $\ell = 2$. Again, an application of Lemma 2.12 rules out these candidates.

Next, let $G = \mathrm{Co}_1$ and $H = 3.\mathrm{Suz}.2$. Here, $H = N_G(\langle z \rangle)$, where z is an element of class $3A$ of G (see [**21**, p. 183]). A computation of class multiplication coefficents shows that there is a conjuate z' of z such that zz' is an involution in Class $2B$. It follows that $L := \langle z, z' \rangle$ is isomorphic to an alternating group A_4, and $N_G(\langle z, z' \rangle) \cong (A_4 \times G_2(4)){:}\,2$ (again, see [**21**, p. 183]). Thus $N_G(L) = (L \times D){:}\,\langle s \rangle$ with $D \cong G_2(4)$, and s an involution normalizing L and D. Now $\langle L, s \rangle \cong S_4$. A computation in S_4 shows that there exists an involution $t \in L$ such that $L = \langle z, z^t \rangle$ and such that t centralizes $N_{\langle L,s \rangle}(\langle z \rangle) \cap N_{\langle L,s \rangle}(\langle z \rangle)^t$. Lemma 2.5 implies that that t centralizes $H \cap H^t$.

Now, take $G = 2.\mathrm{Co}_1$ and $H = 6.\mathrm{Suz}.2$. It suffices to consider faithful characters of G and so we may assume that ℓ is odd. Ruling out the cases $\ell = 7, 11$ and 13 with the known modular character tables for G, we may assume that $\ell = 3$ or 5. Here, $\varphi(1) \leq 660$. The ℓ-modular character tables and decomposition numbers for H are known. If $\ell = 5$, there are just two non-liftable irreducible Brauer characters of degrees at most 660. Both of them have $Z(G)$ in their kernel, so they do not induce to irreducible characters by what we have already proven. Now let $\ell = 3$. Using Lemma 2.12, it is easy to rule out all but two potential examples. The remaining Brauer characters that could possibly induce to irreducible characters have degrees 12. Let $\varphi \in \mathrm{IBr}_3(H)$ with $\varphi(1) = 12$. Then $\bar{\varphi} \neq \varphi$ and $\bar{\varphi} + \varphi$ lifts to an ordinary irreducible character ψ of degree 24. There is exactly one irreducible constituent χ of $\mathrm{Ind}_H^G(\psi)$ with $\chi(1) \geq 12[G{:}\,H]$, and χ occurs with multiplicity 1 in $\mathrm{Ind}_H^G(\psi)$. Moreover, χ is real valued. (In fact, $\chi = \chi_{132}$ in Atlas notation.) Now if $\mathrm{Ind}_H^G(\varphi)$ were irreducible, so would be $\mathrm{Ind}_H^G(\bar{\varphi})$, and both would be containted in the reduction modulo 3 of χ. However, $24[G{:}\,H] > \chi(1)$, so that this possibility does not occur.

Finally, let $G = 2.\mathrm{Co}_1$ and $H = 2 \times \mathrm{Co}_2$ if $\ell = 3$ or 5, and let $G = \mathrm{Co}_1$ and $H = \mathrm{Co}_2$ if $\ell = 2$. The ℓ-modular character tables and decomposition numbers for H are known. Using Lemma 2.12 we can easily rule out H as a block stabilizer. We only comment on the critical cases. Suppose first that ℓ is odd. Then the only candidates arise from the liftable Brauer characters of H of degree 10 395. The lifts of these induce to the irreducible ordinary character χ_{167} of G of largest degree. However, this character is neither irreducible modulo 5 (see [**13**]) nor modulo 3. In each case, the restriction of χ_{167} to the ℓ-regular elements is a $\mathbb{Z}$-linear combination of restricted ordinary characters of

smaller degrees. Now let $\ell = 2$. Then there exists $\varphi \in \mathrm{IBr}_2(H)$ with $\varphi(1) = 748$. The complex conjugate character, $\bar{\varphi}$ is distinct from φ, and φ and $\bar{\varphi}$ are both constituents of the reduction modulo 2 of $\psi \in \mathrm{Irr}(H)$ with $\psi(1) = 1\,771$. There is exactly one irreducible constituent χ of $\mathrm{Ind}_H^G(\psi)$ with $\chi(1) \geq 748 \cdot [G\colon H]$, and χ occurs with multiplicity 1 in $\mathrm{Ind}_H^G(\psi)$. Moreover, χ is real valued. (In fact, $\chi = \chi_{59}$ in Atlas notation.) Now if $\mathrm{Ind}_H^G(\varphi)$ were irreducible, so would be $\mathrm{Ind}_H^G(\bar{\varphi})$, and both would be containted in the reduction modulo 2 of χ. However, $2 \cdot 748 \cdot [G\colon H] > \chi(1)$, so that this possibility does not occur.

This completes our proof for $2.\mathrm{Co}_1$.

$G = J_4$: Only the largest maximal subgroup $H := 2^{11}\colon M_{24}$ of J_4 arises in Step 1. Suppose first that ℓ is odd or $\ell = 0$. Then the two non-trivial orbits of M_{24} on the set of irreducible characters (ordinary or ℓ-modular) of the normal subgroup 2^{11} of H have lengths 759 and 1 288, respectively (see the ordinary character table of H in [**37**]). By [**65**], a non-trivial character of M_{24} has dimension at least 22. But $22 \cdot [J_4\colon H]$ is larger than the largest degree of an ordinary irreducible character of J_4. Suppose then that $\ell = 2$. The 2-modular character table of H is the same as the 2-modular character table of M_{24}. The only characters of H that can possibly induce to an irreducible character of J_4 are the two characters of degree 11. They arise as constitutents of the ordinary irreducible character ψ of M_{24} of degree 23. Inducing ψ from H to J_4 using [**37**] we find

$$\mathrm{Ind}_H^{J_4}(\psi) = \chi_{29} + \chi_{30} + \chi_{45} + \chi_{51}.$$

Each of these characters has degree less than $11 \cdot [J_4\colon H]$. We are done with Lemma 2.12.

$G = 3.\mathrm{Fi}_{24}'$: Step 1 leaves the six largest maximal subgroups as candidates for block stabilizers. If $\ell = 0$, however, only $3 \times \mathrm{Fi}_{23}$ and $3.((3 \times P\Omega_8^+(3)\colon 3)\colon 2)$ remain after Step 2. We now consider these six maximal subgroups in turn, starting with the largest one.

If $G = 3.\mathrm{Fi}_{24}'$ and $H = 3.(3^{1+10}\colon \mathrm{PSU}_5(2)\colon 2)$, then $\varphi(1) = 1$. Hence, φ is liftable. By Lemma 2.11 there are no such examples.

Next, let $G = 3.\mathrm{Fi}_{24}'$ and $H = 3.(3^7.\Omega_7(3))$. Here, $\varphi(1) \leq 3$. The smallest degree of a non-trivial projective character of $\Omega_7(3)$ is, in any characteristic, larger than 3. It follows from Clifford theory, applied to $O_3(H)$, that φ is liftable. Again, Lemma 2.11 rules out this possibility.

In case $G = 3.\mathrm{Fi}_{24}'$ and $H = 3 \times \Omega_{10}^+(2)$, we must have $\varphi(1) \leq 8$, and a similar argument as above disposes of this possibility.

Next, let $G = \mathrm{Fi}_{24}'$ and $H = (3 \times P\Omega_8^+(3)\colon 3)\colon 2$. In this case $H = N_G(\langle z \rangle)$ for an element z in the conjugacy class $3A$ of G (see [**21**, p. 207]). A computation of class multiplication coefficents shows that

there is a conjuate z' of z such that zz' is an involution in Class $2A$. It follows that $L := \langle z, z' \rangle$ is isomorphic to an alternating group A_4, and $N_G(\langle z, z' \rangle) \cong (A_4 \times \Omega_8^+(2){:}\,3){:}\,2$ (again, see [**21**, p. 207]). Thus $N_G(L) = (L \times D){:}\,\langle s \rangle$ with $D \cong \Omega_8^+(2){:}\,3$, and s an involution normalizing L and D. Now $\langle L, s \rangle \cong S_4$. A computation in S_4 shows that there exists an involution $t \in L$ such that $L = \langle z, z^t \rangle$ and such that t centralizes $N_{\langle L,s \rangle}(\langle z \rangle) \cap N_{\langle L,s \rangle}(\langle z \rangle)^t$. Lemma 2.5 implies that that t centralizes $H \cap H^t$ and thus H is not a block stabilizer in G. Finally, Corollary 2.7 excludes $3.H$ as a block stabilizer in $3.G$.

Next, let $G = \mathrm{Fi}_{24}'$ and $H = 2.\mathrm{Fi}_{22}.2$. In this case, $H = C_G(z)$ for a $2A$-element z. A class multiplication coefficient computation shows the existence of a conjugate y of z such that zy lies in class $5A$. Hence there is an element t of order 5 such that $t \in \langle z, z^t \rangle$. Lemma 2.3 and Corollary 2.7 imply that neither H nor its inverse image in $3.G$ are block stabilizers of imprimitive irreducible representations of G, respectively $3.G$.

Let us finally consider the largest maximal subgroup. Suppose first that $G = \mathrm{Fi}_{24}'$ and $H = \mathrm{Fi}_{23}$. Now $H = C_G(z)$ for an element $z \in \mathrm{Fi}_{24}$ lying in conjugacy class $2C$ (see [**21**, p. 207]). There is an element z' conjugate to z such that $t := zz'$ has order 3. Hence $t \in \langle z, z^t \rangle$, and there are no irreducible character of G induced from H by Lemma 2.3. Suppose then that $G = 3.\mathrm{Fi}_{24}'$ and $H = 3 \times \mathrm{Fi}_{23}$. Using GAP, we find exactly the characteristic 0 examples given in Table 1. The character $\zeta_3 \boxtimes \chi_{15}$ of H is of degree 837 200 and is thus of defect 0 for $\ell = 5, 7, 13, 23, 29$. This yields the examples in these characteristics by Lemma 2.11. If $\ell = 11$ or $\ell = 17$, then $\zeta_3 \boxtimes \chi_{15}$ is irreducible modulo ℓ ([**50**, p. 298 and 302]), but χ_{241} is not ([**50**, p. 366 and 378]). The reduction of $\zeta_3 \boxtimes \chi_{15}$ modulo 2 is reducible (see [**53**]), and modulo 3 it is not faithful. Suppose that $\ell \neq 2, 3$. All ℓ-modular character tables of H are known. By applying Lemma 2.12, we find no other examples if ℓ is odd. Since G has no faithful irreducible characters for $\ell = 3$, we may now asume that $\ell = 2$. In this case, the modular character table of H is also known (see [**53**]). Applying Lemma 2.12 once more, only the Brauer characters of degrees 1 494 and 94 588 could possibly induce to irreducible characters of G. To rule out these cases we need a refinement of Lemma 2.12 to Brauer characters. Let φ_1 denote the irreducible Brauer character of H of degree 1 494, and let φ_2 be the Brauer character of H obtained by restricting the ordinary irreducible character of degree 274 482 to the 2-regular elements. Then φ_2 contains the Brauer character of degree 94 588 as a constituent (see [**53**]). In each case, these are faithful Brauer characters lying in one of two

dual blocks. Put $\Phi_i := \mathrm{Ind}_H^G(\varphi_i)$, $i = 1, 2$. We now express Φ_1 and Φ_2 in terms of suitable ordinary characters, restricted to the 2-regular elements of G. The result is given in Table 2. The first column of this table gives the degrees of the restricted ordinary characters, the second and third column give the multiplicities of these restrictions in Φ_1 and Φ_2, respectively. Thus the degree of any modular constituent

TABLE 2. Decomposition of some induced Brauer characters

Degree	Φ_1	Φ_2	Degree	Φ_1	Φ_2
783	−1	−5	203 843 871	−1	−6
64 584	−1	0	216 154 575	1	4
306 153	1	6	216 154 575	1	2
6 724 809	1	6	330 032 934	1	8
19 034 730	−1	−7	1 050 717 096	0	2
25 356 672	0	1	2 801 912 256	0	6
43 779 879	−2	−8	8 993 909 925	0	2
195 019 461	0	7	21 842 179 632	0	2

of $\Phi_1 = \mathrm{Ind}_H^G(1\,494)$ is at most equal to $330\,032\,934 < 1\,494 \times 306\,936$, and the degree of any modular constituent of Φ_2 is at most equal to $21\,842\,179\,632 < 94\,588 \times 306\,936$. Since the second of these factors is the index $[G\colon H]$, the characters $\mathrm{Ind}_H^G(1\,494)$ and $\mathrm{Ind}_H^G(94\,588)$ are reducible. This completes the proof for $3.\mathrm{Fi}_{24}'$.

$G = 2.B$: There are only four maximal subgroups of $2.B$ in the list generated in Step 1, namely $2^2.{}^2E_6(2)\colon 2$, $2.(2^{1+22}\mathrm{Co}_2)$, $2 \times \mathrm{Fi}_{23}$ and $2.(2^{9+16}\colon \mathrm{Sp}_8(2))$. If $H = 2^2.{}^2E_6(2)\colon 2$ or $2.(2^{1+22}\mathrm{Co}_2)$, then the image of H in B is the centralizer of a $2A$ or $2B$ element $z \in B$, respectively (see [**21**, p. 217]). In each case, there is a conjugate z^t such that $\langle z, z^t\rangle \cong S_3$. (This is easily checked using structure constants.) We may thus assume that t is a 3-element in $\langle z, z^t\rangle$. Moreover, as t is a 3-element, there is a 3-element in the preimage of t in $2.B$. Thus, H can not be a block stabilizer by Lemma 2.3 and Corollary 2.7.

If $H = 2 \times \mathrm{Fi}_{23}$, then $\varphi(1) \leq 29$. Therefore φ must be linear. Checking the characteristic 0 case, we see that the corresponding linear character induces to a reducible character.

Finally suppose that $H = 2.(2^{9+16}.\mathrm{Sp}_8(2))$. Then $\varphi(1) < 7$ if $Z(G)$ is in the kernel of φ, and $\varphi(1) < 12$, otherwise. The ordinary character table of the image $\bar{H}$ of H in B is available in GAP. (It has been computed by Eamonn O'Brien.) We find that $\bar{H}$ is perfect. Moreover,

GAP shows that $\bar{H}$ is not a subgroup of $2.B$. Thus H is perfect as well. Suppose first that $\ell = 2$. In this case every irreducible 2-modular character of H is a 2-modular character of $\mathrm{Sp}_8(2)$. However, the smallest degree of a non-trivial 2-modular character of $\mathrm{Sp}_8(2)$ equals 8. Thus there is no imprimitive irreducible 2-modular character of G, since this would have $Z(G)$ in its kernel. Suppose then that $\ell > 2$ or $\ell = 0$ and let φ be an irreducible character of H of degree smaller than 12. Put $N := O_2(H)$ and let λ be an irreducible constituent of $\mathrm{Res}_N^H(\varphi)$. If λ were not invariant in H, the degree of φ would be at least equal to the index of a proper subgroup of $\mathrm{Sp}_8(2)$, hence at least equal to 120. Thus λ is invariant in H. Since the Schur multiplier of $\mathrm{Sp}_8(2)$ is trivial, and since a non-trivial character of $\mathrm{Sp}_8(2)$ has degree at least 35, we conclude that φ is an extension of λ. Suppose now that $\ell \neq 0$. The restriction of φ to N equals λ, which lifts to characteristic 0, since N is a ℓ'-group. Thus φ also lifts to an ordinary character ψ, say, namely to the extension to H of the lift of $\mathrm{Res}_N^H(\varphi)$. But no ordinary character degree of $2.B$ is divisible by the index of H in G. In particular, $\mathrm{Ind}_H^G(\psi)$ is reducible. Then $\mathrm{Ind}_H^G(\varphi)$ is reducible as well by Lemma 2.11(1). This completes the proof for $2.B$.

$G = M$: It follows from the known information on the maximal subgroups of M (see [**109**]), that the only maximal subgroup which is large enough is $2.B$. Now $H = 2.B$ is the centralizer of a $2A$-element $z \in M$ (see [**21**, p. 234]). Using structure constants one checks that there is a conjugate z' of z such that $t := zz'$ has order 3. Hence $t \in \langle z, z^t \rangle$ and thus, by Lemma 2.3, the subgroup $2.B$ does not give rise to imprimitive irreducible characters of M. $\diamond$

CHAPTER 4

Alternating Groups

In this chapter G is an alternating group or a the double cover of an alternating group (or possibly the triple or sixfold cover for A_6 or A_7). The results presented here are known, but included for the sake of completeness. Again, K denotes an algebraically closed field of characteristic $\ell \geq 0$, and the results are formulated in terms of K-characters.

We first consider the groups A_n for small values of n. The examples are given in Table 1, where we use similar conventions as for Table 1.

THEOREM 4.1. *Let G be a covering group of A_n with $5 \leq n \leq 9$. Then the irreducible imprimitive K-characters of G are exactly those described in Table 1.*

PROOF. The result is easily obtained with the help of GAP. These groups have also been treated in [**30**] and [**90**] in case $\mathrm{char}(K) = 0$. ⋄

If n is large enough, the possible maximal block stabilizers are rather restricted.

PROPOSITION 4.2. [Nett-Noeske [**90**]] *Suppose that $n \geq 10$ and let $G = 2.A_n$ denote the twofold cover of A_n. Let $H \leq G$ be the inverse image of a maximal, transitive subgroup of A_n. If H is the block stabilizer of an irreducible imprimitive non-trivial K-character, then $H/Z(G) = A_{n-1}$ or $n = 2m$ is even and $H/Z(G) = (S_m \wr S_2) \cap A_n$.*

The next result gives all generic examples in case of $\mathrm{char}(K) = 0$.

THEOREM 4.3. [Djoković-Malzan, Nett-Noeske, Schur] *Suppose that* $\mathrm{char}(K) = 0$. *Then the irreducible imprimitive K-characters of A_n or $2.A_n$, $n \geq 5$ are exactly those given in Table 2.*

The group E in the middle column of this table is a group of order 4*; it is elementary abelian, if m is even, and cyclic, otherwise. The characters* 1_3 *and* 1_4 *in the same row are characters of degree* 1 *which have $A_m \times A_m$ in their kernel, but not $(S_m \times S_m) \cap A_m$.*

The Frobenius-Schur indicators in this table are computed as follows:

$$\iota_m = \begin{cases} +, & \text{if } m(m-1)/2 \text{ is even and} \\ \circ, & \text{if } m(m-1)/2 \text{ is odd;} \end{cases}$$

TABLE 1. Small alternating group examples

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes
A_5	A_4	ζ_3 $\bar{\zeta}_3$	$\circ$	1	χ_5	$+$	5	$0, 5$
$2.A_5$	$5:4$	ζ_4 $\bar{\zeta}_4$	$\circ$	1	χ_9	$-$	6	$0, 3$
A_6	$3^2:4$	ζ_4 $\bar{\zeta}_4$	$\circ$	1	χ_7	$+$	10	$0, 5$
$2.A_6$	$3^2:8$	ζ_8 $\bar{\zeta}_8$	$\circ$	1	χ_{12}	$-$	10	$0, 5$
		ζ_8^3 $\bar{\zeta}_8^3$	$\circ$	1	χ_{13}	$-$	10	$0, 5$
$3.A_6$	$3 \times A_5$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{16}	$\circ 2$	6	$0, 5$
$6.A_6$	$3 \times 2.A_5$	$\zeta_3 \boxtimes 2_1$	$\circ 2$	2	χ_{21}	$\circ 2$	12	0
		$\zeta_3 \boxtimes 2_2$	$\circ 2$	2	χ_{22}	$\circ 2$	12	0
$3.A_7$	$3.A_6$	ψ_{14} ψ_{15}	$\circ 2$	3	χ_{21}	$\circ 2$	21	$0, 7$
	$3 \times \mathrm{SL}_3(2)$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{19}	$\circ 2$	15	$0, 5$
	$3 \times S_5$	$\zeta_3 \boxtimes \zeta_2$	$\circ 2$	1	χ_{20}	$\circ 2$	21	$0, 5, 7$
A_8	$2^3:\mathrm{SL}_3(2)$	ψ_2	$\circ 2$	3	χ_{10}	$\circ 2$	45	$0, 3, 5$
		3	$+$	3	φ_{10}	$+$	45	7
	$2^4:(S_3 \times S_3)$	$\zeta_2 \boxtimes 1$ $1 \boxtimes \zeta_2$	$+$	1	χ_9	$+$	35	$0, 3, 5, 7$
		$\zeta_2 \boxtimes 2$ $2 \boxtimes \zeta_2$	$+$	2	χ_{14}	$+$	70	$0, 5, 7$
A_9	$\mathrm{SL}_2(8):3$	ζ_3 $\bar{\zeta}_3$	$\circ$	1	χ_{14}	$+$	120	$0, 5$

$$\kappa_m = \begin{cases} +, & \text{if } m \text{ is even,} \\ \circ, & \text{if } m \text{ is odd;} \end{cases}$$

$$\lambda_m = \begin{cases} +, & \text{if } m(m-1)/2 \equiv (0 \bmod 8), \\ -, & \text{if } m(m-1)/2 \equiv (4 \bmod 8), \\ \circ, & \text{if } m(m-1)/2 \equiv (2 \bmod 4); \end{cases}$$

$$\nu_m = \begin{cases} +, & \text{if } m(m-1)/2 \equiv (0, 6 \bmod 8), \\ -, & \text{if } m(m-1)/2 \equiv (2, 4 \bmod 8). \end{cases}$$

PROOF. The result for the simple groups is contained in [**30**], the result for the covering groups in [**90**]. The degrees of the characters involved can be computed from the hook formulae ([**64**, Theorem 2.3.31]

and [**55**, Theorem 10.7]). The indicators of the characters are determined from [**64**, Theorem 2.5.13] and [**95**, Sätze IV, V]. $\diamond$

TABLE 2. Generic alternating group examples (characteristic 0)

G	$A_{m^2+1}, m \geq 2$	$A_{2m}, m \geq 3$	$2.A_n, n = \frac{m(m+1)+2}{2}, \frac{m(m-1)}{2}$ even
H	A_{m^2}	$(A_m \times A_m) : E$	$2.A_{n-1}$
ψ	$[m^m]_1$ $[m^m]_2$	1_3 1_4	$< m, m-1, \ldots, 2, 1 >_1$ $< m, m-1, \ldots, 2, 1 >_2$
ind	ι_m	κ_m	λ_m
$\psi(1)$	$\frac{(m^2)!}{2m^m \prod_{i=1}^{m-1} [i(2m-i)]^i}$	1	$2^{m(m-1)/4-1} \cdot \frac{(n-1)!}{m!} \prod_{i=1}^{m-1} \frac{(i+1)!}{(2i+1)!}$
χ	$[m+1, m^{m-1}]$	$[m+1, 1^{m-1}]$	$< m+1, m-1, \ldots, 2, 1 >$
ind	$+$	$+$	ν_m
$\chi(1)$	$(m^2+1)\psi(1)$	$\binom{2m-1}{m}$	$n \cdot \psi(1)$

CHAPTER 5

Exceptional Schur multipliers, exceptional isomorphisms and the Tits group

In this chapter K is an algebraically closed field of characteristic $\ell \geq 0$ and we use, once more, the language of K-characters. We determine the irreducible imprimitive K-characters for all quasisimple groups G of Lie type, such that $G/Z(G)$ has an exceptional Schur multiplier or such that $G/Z(G)$ is isomorphic to a simple group of Lie type of a different characteristic, with the only restriction that $\mathrm{char}(K) = 0$ if $G = 6.{}^2E_6(2)$. For the sake of uniform statements in later chapters, we also consider the Tits ${}^2F_4(2)'$ group here. We begin with the latter.

PROPOSITION 5.1. *The Tits group* ${}^2F_4(2)'$ *does not have any imprimitive irreducible* K*-characters.*

PROOF. The (modular) character tables of the Tits group and of all of its maximal subgroups are available in GAP. It is thus a routine task to check our assertion. $\diamond$

The simple groups of Lie type with an exceptional Schur multiplier are listed in Table 1 (cf. [**42**, Table 6.1.3]). There are two entries for the groups $\mathrm{PSL}_2(9)$ and $\mathrm{Sp}_4(2)'$, although these groups are isomorphic. The reason is that each of them has an exceptional Schur multiplier, one as a group of Lie type of characteristic 3, the other as a group of Lie type of characteristic 2. On the other hand we have not listed the group $\mathrm{PSp}_4(3)$ isomorphic to $\mathrm{SU}_4(2)$, since $\mathrm{PSp}_4(3)$ does not have an exceptional multiplier as group of Lie type of characteristic 3. For the same reason we have not listed $\mathrm{PSL}_2(5)$ nor $\mathrm{PSL}_2(7)$.

TABLE 1. The simple groups of Lie type with exceptional Schur multiplier

$\mathrm{SL}_2(4)$	$\mathrm{PSL}_2(9)$	$\mathrm{Sp}_4(2)'$	$\mathrm{SL}_3(2)$	$\mathrm{PSL}_3(4)$	$\mathrm{SL}_4(2)$
$\mathrm{SU}_4(2)$	$\mathrm{PSU}_4(3)$	$\mathrm{PSU}_6(2)$	$\mathrm{PSp}_4(2)'$	$\mathrm{PSp}_6(2)$	$\Omega_7(3)$
${}^2B_2(8)$	$\Omega_8^+(2)$	$G_2(3)$	$G_2(4)$	$F_4(2)$	${}^2E_6(2)$

There are exactly six exceptional isomorphisms between simple groups of Lie type of different characteristics (see [**42**, Theorem 2.2.10]). We give these isomorphisms in Table 2. Of these groups, only ${}^2G_2(3)' \cong \mathrm{SL}_2(8)$ and $G_2(2)' \cong \mathrm{SU}_3(3)$ do not occur in Table 1.

TABLE 2. Exceptional isomorphisms

$\mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5)$,	$\mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$,
$\mathrm{PSL}_2(9) \cong \mathrm{Sp}_4(2)'$,	${}^2G_2(3)' \cong \mathrm{SL}_2(8)$,
$G_2(2)' \cong \mathrm{SU}_3(3)$,	$\mathrm{SU}_4(2) \cong \mathrm{PSp}_4(3)$.

The following theorem is the main result of this chapter.

THEOREM 5.2. *Let G be a quasisimple group such that $G/Z(G)$ is one of the groups occurring in Tables 1 and 2, and let H be a maximal subgroup of G. In case $G/Z(G) = {}^2E_6(2)$ and $H/Z(G)$ is a parabolic subgroup of $G/Z(G)$, assume that* $\mathrm{char}(K) = 0$.

If χ is an irreducible faithful K-character of G such that $\chi = \mathrm{Ind}_H^G(\psi)$ for some K-character ψ of H, then χ is as listed in one of the Tables 3 - 9. In the first two of these tables, $H/Z(G)$ is a non-parabolic subgroup of $G/Z(G)$, in the last five of these tables, $H/Z(G)$ is parabolic.

The proof of Theorem 5.2 is given in Section 5.2 below. Since we only consider faithful irreducible characters of G, the center $Z(G)$ is cyclic in all cases, even if the Schur multiplier of the simple group $G/Z(G)$ is non-cyclic. Also, if $\mathrm{Ind}_H^G(\psi)$ is faithful, the restriction of ψ to $Z(G) \leq H$ is faithful as well.

5.1. Description of the tables

We employ the same conventions as in Table 1 for the sporadic groups. There are a couple of additions to the notations used there. If H is of the form $L.2$ and ψ is an extension of an irreducible character ψ_i of L, where i is the number in the compound character table of the Atlas, we write $\psi = \psi_i'$ or ψ_i''. An example occurs in Table 3, in the entry for $G = 2.\mathrm{Sp}_6(2)$ and $H = 2.\mathrm{SU}_4(2){:}\,2$. Here, the group $L := 2.\mathrm{SU}_4(2)$ has a faithful irreducible character of degree 20 and Frobenius-Schur indicator $-$, denoted χ_{23} in [**21**]. The two extensions of χ_{23} to H are denoted by ψ_{23}' and ψ_{23}'' in Table 3. A similar convention is used for characters which are just denoted by their degrees. A further addition concerns the tables with the parabolic examples. Suppose

that G is of Lie type of characteristic ℓ and that H is a parabolic subgroup of G. Then H is of the form $U{:}L$, with $U = O_\ell(H)$, the "unipotent radical" of H. In this situation U is in the kernel of ψ by Proposition 7.1, and we give the Atlas name of ψ as a character of L. Consider, for example, the entry for $G = \mathrm{SL}_4(2)$ and $H = 2^3{:}\mathrm{SL}_3(2)$ in Table 5. Here, the character ψ_2 denotes the irreducible character of the simple group $\mathrm{SL}_3(2)$ with Atlas name χ_2.

Here, we also have an additional column headed "Nts" which contains further information of various kind, which we are now going to describe in more detail.

1. Let p be the defining characteristic of G. There is an isomorphism of G with a group of Lie type G' of characteristic $p' \neq p$, which maps H to a parabolic subgroup of G'. (H is a parabolic subgroup in the wrong characteristic.)
2. There are two K-characters $\psi_1 \neq \psi_2$ of H with $\mathrm{Ind}_H^G(\psi_1) = \chi = \mathrm{Ind}_H^G(\psi_2)$.
3. This is an example with char $(K) = \ell = p$, where p is the defining characteristic of G. It also occurs in the result of Seitz cited above (Theorem 1.2).
4. There is more than one maximal block stabilizer for χ (up to conjugation in G).
5. This is an example where χ is not liftable to characteristic 0.
6. This is a non-canonical "parabolic" example, i.e., $H/Z(H)$ is a parabolic subgroup of $G/Z(G)$, but G itself is not a group of Lie type.

f_ℓ. Let p be the defining characteristic of G. There is an isomorphism of G with a group of Lie type G' of characteristic $\ell \neq p$. (A defining, but wrong characteristic example). It also occurs in the result of Seitz cited above (Theorem 1.2).

†. There is an outer automorphism of G of order 2 transporting the triple (H, ψ, χ) to the triple (H', ψ', χ') such that H and H' are not conjugate in G and $\chi \neq \chi'$. We have not included a row for (H', ψ', χ').

†′. Similar to †, but with an outer automorphism of order 3.

Additional explanations of various details are contained in Remark 5.3 below.

Remark 5.3. *Table 3.* Entry for $2.\mathrm{SU}_4(2)$: The two characters 2_2 and 2_3 of $H = 3^3{:}\mathrm{GL}_2(3)$ denote the two faithful characters of $\mathrm{GL}_2(3)$ of degree 2, inflated to H.

Table 4. (1) Entries for $3_1.\mathrm{SU}_4(3) = 12_1.\mathrm{PSU}_4(3)$, $3_2.\mathrm{SU}_4(3) = 12_2.\mathrm{PSU}_4(3)$: The subgroup $4{\circ}\mathrm{Sp}_4(3)$, the central product of $Z(\mathrm{SU}_4(3))$

with $\mathrm{Sp}_4(3)$ (see [**69**, Table 3.5 B]) is isoclinic to $2\times 2.\mathrm{SU}_4(2)$. The characters ψ'_{21} and ψ'_{22} denote extensions of the two characters of degree 4 of $\mathrm{Sp}_4(3)$ to H. There are two conjugacy classes of subgroups isomorphic to $4\circ \mathrm{Sp}_4(3)$ in each of $3_1.\mathrm{SU}_4(3)$ and $3_2.\mathrm{SU}_4(3)$. In $3_1.\mathrm{SU}_4(3)$, only one of these conjugacy classes contains block stabilizers. In $3_2.\mathrm{SU}_4(3)$ the two conjugacy classes are swapped by an outer automorphism of $3_2.\mathrm{SU}_4(3)$ (see [**21**, pp. 52–53]), and so both conjugacy classes contain block stabilizers.

(2) At various places in Table 4 the following phenomenon occurs. There is an orbit of length four of Galois conjugate irreducible characters of H, but only two of the four induce to irreducible characters of G. Consider, for example, the entry for $G = 3_2.\mathrm{PSU}_4(3)$ and $H = 3\times \mathrm{SU}_4(2)$. There are four irreducible characters of degree 5 which restrict to faithful characters of $Z(G)$, namely $\zeta_3\boxtimes\psi_2$, $\bar\zeta_3\boxtimes\psi_2$, $\zeta_3\boxtimes\psi_3$, and $\bar\zeta_3\boxtimes\psi_3$, but only two of these, a pair of complex conjugates, induce to irreducible characters of G. This happens exactly in the following cases, described in the form (G, H, ψ), where ψ is one of the four characters in the Galois orbit.

G	H	ψ
$3_2.\mathrm{PSU}_4(3)$	$3\times \mathrm{SU}_4(2)$	$\zeta_3\boxtimes\psi_2$
$6_2.\mathrm{PSU}_4(3)$	$6\times \mathrm{SU}_4(2)$	$\zeta_6\boxtimes\psi_2$
$3.\Omega_7(3)$	$3\times \mathrm{PSL}_4(3){:}2_2$	$\zeta_3\boxtimes\zeta_2$
$2.\mathrm{SU}_6(2)$	$6\times \mathrm{SU}_5(2)$	$\zeta_6\boxtimes\psi_3$

Table 5. (1) Entries for $\mathrm{PSL}_3(4)$ and $\mathrm{SL}_3(4)$: The character denoted by 3 is the inflation to H of the reduction modulo 5 of the two ordinary characters ψ_2 and ψ_3 of A_5.

(2) Entry for $4_2.\mathrm{SL}_3(4) = 12_2.\mathrm{PSL}_3(4)$: The character denoted by 4_2 is one of two faithful characters of $4.2^4{:}A_5$ of degree 4, inflated to H.

Table 6. (1) Entries for $\mathrm{PSU}_4(3)$ and $G_2(3)$: The two characters 2_2 and 2_3 of $H = 3_+^{1+4}{:}2.S_4$, respectively $H = (3_+^{1+2}\times 3^2){:}2.S_4$ denote the two faithful characters of the Levi subgroup $L = 2.S_4$ of H of degree 2, inflated to H.

(2) Entry for $2.\mathrm{PSU}_4(3)$: The two characters 2_4 and 2_5 of $H = 3_+^{1+4}{:}2.(2.S_4)$ denote two irreducible characters of the Levi subgroup $L = 2.(2.S_4)$ of H of degree 2, inflated to H. These two characters are determined by the fact that $Z(H)$ is not in their kernel and that they are not rational.

(3) Entry for $\mathrm{SU}_4(3) = 4.\mathrm{PSU}_4(3)$: The two characters denoted by 1_3 and 1_4 are linear characters of order 8. The characters denoted by 3_4 and 3_5 denote two faithful characters of the Levi subgroup $L = 4.(2.S_4)$ of H of degree 3, inflated to H.

Table 7. (1) Entries for $\Omega_7(3)$: The two characters 2_5 and 2_6 of $H = 3_+^{1+6}{:}(2.A_4 \times A_4).2$ denote the two non-rational irreducible characters of the Levi subgroup $L = (2.A_4 \times A_4).2$ of H of degree 2, inflated to H. Moreover, 6_3 and 6_4 are the two faithful irreducible characters of degree 6 of L inflated to H.

(2) Entries for $2.\Omega_7(3)$: The two characters 2_7 and 2_8 of H denote the two non-rational irreducible characters of the Levi subgroup $L = 2.(2.A_4 \times A_4).2$ of H of degree 2, which do not have $Z(H)$ in their kernel, inflated to H. Moreover, 6_5 and 6_6 are the two non-rational irreducible characters of degree 6 of L, which do not have $Z(H)$ in their kernel, inflated to H.

TABLE 3. Non-parabolic examples

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes	Nts
$2.\mathrm{SL}_2(4)$	$5:4$	ζ_4 $\bar{\zeta}_4$	$\circ$	1	χ_9	$-$	6	$0, 3$	$1, 2$
$\mathrm{SL}_3(2)$	$7:3$	ζ_3 $\bar{\zeta}_3$	$\circ$	1	χ_6	$+$	8	$0, 2$	$1, 2, 3$
$2.\mathrm{SL}_3(2)$	$7:6$	ζ_6 $\bar{\zeta}_6$	$\circ$	1	χ_{11}	$-$	8	0	$1, 2$
$3.\mathrm{PSL}_2(9)$	$3 \times A_5$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{16}	$\circ 2$	6	$0, 5$	4
$3.\mathrm{SL}_2(9)$	$3 \times 2.A_5$	$\zeta_3 \boxtimes \psi_6$	$\circ 2$	2	χ_{21}	$\circ 2$	12	0	4
		$\zeta_3 \boxtimes \psi_7$	$\circ 2$	2	χ_{22}	$\circ 2$	12	0	4
$\mathrm{Sp}_4(2)'$	$3^2:4$	ζ_4 $\bar{\zeta}_4$	$\circ$	1	χ_7	$+$	10	$0, 5$	$1, 2$
$2.\mathrm{Sp}_4(2)'$	$3^2:8$	ζ_8 $\bar{\zeta}_8$	$\circ$	1	χ_{12}	$-$	10	$0, 5$	$1, 2$
		ζ_8^3 $\bar{\zeta}_8^3$	$\circ$	1	χ_{13}	$-$	10	$0, 5$	$1, 2$
$3.\mathrm{Sp}_4(2)'$	$3 \times A_5$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{16}	$\circ 2$	6	$0, 5$	4
$6.\mathrm{Sp}_4(2)'$	$3 \times 2.A_5$	$\zeta_3 \boxtimes \psi_6$	$\circ 2$	2	χ_{21}	$\circ 2$	12	0	4
		$\zeta_3 \boxtimes \psi_7$	$\circ 2$	2	χ_{22}	$\circ 2$	12	0	4
${}^2G_2(3)'$	$2^3:7$	ζ_7 $\bar{\zeta}_7$	$\circ$	1	χ_7	$+$	9	$0, 3$	$1, 2$
		ζ_7^3 $\bar{\zeta}_7^3$	$\circ$	1	χ_8	$+$	9	$0, 3$	$1, 2$
		ζ_7^2 $\bar{\zeta}_7^2$	$\circ$	1	χ_9	$+$	9	$0, 3$	$1, 2$
$G_2(2)'$	$3_+^{1+2}:8$	ζ_8 ζ_8^5	$\circ$	1	χ_{11}	$\circ$	28	$0, 7$	$1, 2$
		ζ_8 $\bar{\zeta}_8^5$	$\circ$	1	χ_{12}	$\circ$	28	$0, 7$	$1, 2$
$2.\mathrm{SL}_4(2)$	$2.A_7$	6_1 6_2	$-$	6	φ_{15}	$-$	48	3	$2, 5$
$4_1.\mathrm{PSL}_3(4)$	$4 \times A_6$	$\zeta_4 \boxtimes 1$	$\circ 2$	1	χ_{21}	$\circ 2$	56	$0, 5, 7$	4
$4_1.\mathrm{SL}_3(4)$	$12 \times \mathrm{PSL}_2(7)$	$\zeta_{12} \boxtimes 1$	$\circ 4$	1	χ_{52}	$\circ 4$	120	$0, 5$	4
$2.\mathrm{SU}_4(2)$	$3^3:\mathrm{GL}_2(3)$	2_2 2_3	$\circ$	2	χ_{34}	$-$	80	$0, 5$	$1, 2$
$2.\mathrm{Sp}_6(2)$	$2.\mathrm{SU}_4(2):2$	ψ'_{23} ψ''_{23}	$\circ$	20	χ_{42}	$+$	560	$0, 5, 7$	2
	$2 \times \mathrm{SU}_3(3):2$	$\zeta_2 \boxtimes \zeta_2$	$+$	1	χ_{37}	$+$	120	$0, 5, 7$	
		$\zeta_2 \boxtimes \psi'_2$ $\zeta_2 \boxtimes \psi''_2$	$\circ$	6	χ_{43}	$+$	720	$0, 5$	2

Table 4. Non-parabolic examples (continued)

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes	Nts
$3_1.\mathrm{PSU}_4(3)$	$3 \times \mathrm{SU}_4(2)$	$\zeta_3 \boxtimes \psi_2$ $\zeta_3 \boxtimes \psi_3$	$\circ 2$	5	χ_{68}	$\circ 2$	630	$0, 5, 7$	$2, 4$
$3_2.\mathrm{PSU}_4(3)$	$3 \times \mathrm{SU}_4(2)$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{102}	$\circ 2$	126	$0, 5, 7$	4
		$\zeta_3 \boxtimes \psi_2$	$\circ 2$	5	χ_{107}	$\circ 2$	630	$0, 5, 7$	4
$6_1.\mathrm{PSU}_4(3)$	$6 \times \mathrm{SU}_4(2)$	$\zeta_6 \boxtimes 1$	$\circ 2$	1	χ_{75}	$\circ 2$	126	$0, 5, 7$	4
		$\zeta_6 \boxtimes \psi_2$ $\zeta_6 \boxtimes \psi_3$	$\circ 2$	5	χ_{82}	$\circ 2$	630	$0, 5, 7$	$2, 4$
$6_2.\mathrm{PSU}_4(3)$	$6 \times \mathrm{SU}_4(2)$	$\zeta_6 \boxtimes 1$	$\circ 2$	1	χ_{113}	$\circ 2$	126	$0, 5, 7$	†
		$\zeta_6 \boxtimes \psi_2$	$\circ 2$	5	χ_{121}	$\circ 2$	630	$0, 5, 7$	4
$3_1.\mathrm{SU}_4(3)$	$3 \times (4 \circ \mathrm{Sp}_4(3))$	$\zeta_3 \boxtimes \psi'_{21}$ $\zeta_3 \boxtimes \psi'_{22}$	$\circ 4$	4	χ_{95}	$\circ 4$	504	$0, 7$	2
$3_2.\mathrm{SU}_4(3)$	$3 \times (4 \circ \mathrm{Sp}_4(3))$	$\zeta_3 \boxtimes \psi'_{21}$	$\circ 4$	4	χ_{128}	$\circ 4$	504	$0, 7$	4
		$\zeta_3 \boxtimes \psi'_{22}$	$\circ 4$	4	χ_{129}	$\circ 4$	504	$0, 7$	4
$3.G_2(3)$	$3 \times (\mathrm{SU}_3(3){:}\,2)$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{28}	$\circ 2$	351	$0, 13$	4
		$\zeta_3 \boxtimes \zeta_2$	$\circ 2$	1	χ_{29}	$\circ 2$	351	$0, 7, 13$	†
	$3 \times (\mathrm{SL}_3(3){:}\,2)$	$\zeta_3 \boxtimes \zeta_2$	$\circ 2$	1	χ_{31}	$\circ 2$	378	$0, 7, 13$	†
$2.G_2(4)$	$(2 \times \mathrm{SU}_3(4)).2$	ζ_4 $\bar{\zeta}_4$	$\circ$	1	χ_{43}	$-$	2016	$0, 3, 7$	2
$3.\Omega_7(3)$	$3 \times \mathrm{PSL}_4(3){:}\,2_2$	$\zeta_3 \boxtimes \zeta_2$	$\circ 2$	1	χ_{92}	$\circ 2$	378	$0, 5, 7, 13$	
$6.\Omega_7(3)$	$3 \times \mathrm{SL}_4(3){:}\,2_2$	$\zeta_3 \boxtimes \psi'_{30}$ $\zeta_3 \boxtimes \psi''_{30}$	$\circ 2$	40	χ_{127}	$\circ 2$	15 120	$0, 5, 7$	2
	$2 \times 3.G_2(3)$	$\zeta_2 \boxtimes \psi_{25}$	$\circ 2$	27	χ_{134}	$\circ 2$	29160	$0, 5$	4
$2.\mathrm{SU}_6(2)$	$6 \times \mathrm{SU}_5(2)$	$\zeta_6 \boxtimes 1$	$\circ 2$	1	χ_{122}	$\circ 2$	672	$0, 5, 7, 11$	
		$\zeta_6 \boxtimes \psi_3$	$\circ 2$	11	χ_{130}	$\circ 2$	7392	$0, 5, 7, 11$	

TABLE 5. Parabolic examples

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes	Nts
$\mathrm{SL}_2(4)$	$2^2\!:\!3$	ζ_3 $\bar{\zeta}_3$	$\circ$	1	χ_5	$+$	5	$0, 5$	$2, f_5$
$\mathrm{SL}_3(2)$	$2^2\!:\!\mathrm{SL}_2(2)$	ζ_2	$+$	1	χ_5	$+$	7	$0, 3, 7$	$4, f_7$
$\mathrm{PSL}_2(9)$	$3^2\!:\!4$	ζ_4 $\bar{\zeta}_4$	$\circ$	1	χ_7	$+$	10	$0, 5$	2
$\mathrm{SL}_2(9)$	$3^2\!:\!8$	ζ_8 $\bar{\zeta}_8$	$\circ$	1	χ_{12}	$-$	10	$0, 5$	2
		ζ_8^3 $\bar{\zeta}_8^3$	$\circ$	1	χ_{13}	$-$	10	$0, 5$	2
$\mathrm{SL}_4(2)$	$2^3\!:\!\mathrm{SL}_3(2)$	ψ_2	$\circ 2$	3	χ_{10}	$\circ 2$	45	$0, 3, 5$	4
		3	$+$	3	φ_{10}	$+$	45	7	4
	$2^4\!:\!(\mathrm{SL}_2(2) \times \mathrm{SL}_2(2))$	$\zeta_2 \boxtimes 1$ $1 \boxtimes \zeta_2$	$+$	1	χ_9	$+$	35	$0, 3, 5, 7$	2
		$\zeta_2 \boxtimes 2$ $2 \boxtimes \zeta_2$	$+$	2	χ_{14}	$+$	70	$0, 5, 7$	2
$\mathrm{PSL}_3(4)$	$2^4\!:\!A_5$	ψ_2	$+$	3	χ_8	$+$	63	$0, 3, 7$	4
		ψ_3	$+$	3	χ_9	$+$	63	$0, 3, 7$	4
		3	$+$	3	φ_8	$+$	63	5	4
$\mathrm{SL}_3(4)$	$2^4\!:\!(3 \times A_5)$	$\zeta_3 \boxtimes 1$	$\circ 2$	1	χ_{35}	$\circ 2$	21	$0, 5, 7$	4
		$\zeta_3 \boxtimes \psi_2$	$\circ 2$	3	χ_{38}	$\circ 2$	63	$0, 7$	4
		$\zeta_3 \boxtimes \psi_3$	$\circ 2$	3	χ_{39}	$\circ 2$	63	$0, 7$	4
		$\zeta_3 \boxtimes 3$	$\circ 2$	3	φ_{30}	$\circ 2$	63	5	4
		$\zeta_3 \boxtimes \psi_4$	$\circ 2$	4	χ_{40}	$\circ 2$	84	$0, 7$	4
$4_2.\mathrm{SL}_3(4)$	$3 \times (4.2^4\!:\!A_5)$	$\zeta_3 \boxtimes 4_2$	$\circ 4$	4	χ_{58}	$\circ 4$	84	$0, 7$	$4, 6$

Table 6. Parabolic examples (continued)

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes	Nts
$\mathrm{SU}_4(2)$	$2^4\!:\!A_5$	ψ_2 ψ_3	$+$	3	χ_{20}	$+$	81	$0, 3$	$2, f_3$
${}^2B_2(8)$	$2^{3+3}\!:\!7$	ζ_7 $\bar{\zeta}_7$	$\circ$	1	χ_8	$+$	65	$0, 5, 13$	2
		ζ_7^3 $\bar{\zeta}_7^3$	$\circ$	1	χ_9	$+$	65	$0, 5, 13$	2
		ζ_7^2 $\bar{\zeta}_7^2$	$\circ$	1	χ_{10}	$+$	65	$0, 5, 13$	2
$\mathrm{Sp}_6(2)$	$2^6\!:\!\mathrm{SL}_3(2)$	ψ_2 ψ_3	$\circ$	3	χ_{28}	$+$	405	$0, 3, 5$	2
$\mathrm{PSU}_4(3)$	$3^4\!:\!\mathrm{PSL}_2(9)$	ψ_4 ψ_5	$+$	8	χ_{20}	$+$	896	$0, 2, 7$	2
	$3_+^{1+4}\!:\!2.S_4$	2_2 2_3	$-$	2	χ_{16}	$-$	560	$0, 5, 7$	2
$2.\mathrm{PSU}_4(3)$	$3^4\!:\!(2 \times \mathrm{PSL}_2(9))$	$\zeta_2 \boxtimes \psi_4$ $\zeta_2 \boxtimes \psi_5$	$+$	8	χ_{39}	$+$	896	$0, 7$	2
	$3_+^{1+4}\!:\!2.(2.S_4)$	2_4 2_5	$\circ$	2	χ_{35}	$+$	560	$0, 5, 7$	2
$\mathrm{SU}_4(3)$	$3^4\!:\!2.(2 \times \mathrm{PSL}_2(9))$	ψ'_{10} ψ'_{11}	$\circ 2$	8	χ_{65}	$\circ 2$	896	$0, 7$	2
	$3_+^{1+4}\!:\!4.(2.S_4)$	1_3 1_4	$\circ 2$	1	χ_{47}	$\circ 2$	280	$0, 5, 7$	2
		3_4 3_5	$\circ 2$	3	χ_{54}	$\circ 2$	840	$0, 5, 7$	2
$G_2(3)$	$(3_+^{1+2} \times 3^2)\!:\!2.S_4$	2_2 2_3	$\circ$	2	χ_{19}	$+$	728	$0, 7, 13$	$2, \dagger$
$\Omega_8^+(2)$	$2^6\!:\!A_8$	ψ_6 ψ_7	$\circ$	21	χ_{47}	$+$	2 835	$0, 7$	$2, \dagger'$
		ψ_{10} ψ_{11}	$\circ$	45	χ_{53}	$+$	6 075	$0, 3, 5$	$2, 4$

TABLE 7. Parabolic examples (continued)

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes	Nts
$G_2(4)$	$2^{2+8}:(3 \times A_5)$	$\zeta_3 \boxtimes \psi_2$ $\bar{\zeta}_3 \boxtimes \psi_2$	$\circ$	3	χ_{26}	+	4 095	$0, 5, 7, 13$	2
		$\zeta_3 \boxtimes \psi_3$ $\bar{\zeta}_3 \boxtimes \psi_3$	$\circ$	3	χ_{27}	+	4 095	$0, 5, 7, 13$	2
	$2^{4+6}:(A_5 \times 3)$	$1 \boxtimes \zeta_3$ $1 \boxtimes \bar{\zeta}_3$	$\circ$	1	χ_{16}	+	1 365	$0, 5, 7, 13$	2
		$\psi_2 \boxtimes \zeta_3$ $\psi_2 \boxtimes \bar{\zeta}_3$	$\circ$	3	χ_{24}	+	4 095	$0, 5, 7, 13$	2
		$\psi_3 \boxtimes \zeta_3$ $\psi_3 \boxtimes \bar{\zeta}_3$	$\circ$	3	χ_{25}	+	4 095	$0, 5, 7, 13$	2
		$\psi_4 \boxtimes \zeta_3$ $\psi_4 \boxtimes \bar{\zeta}_3$	$\circ$	4	χ_{32}	+	5 460	$0, 7, 13$	2
$\Omega_7(3)$	$3^{3+3}:\mathrm{SL}_3(3)$	ψ_4 ψ_5	$\circ$	16	χ_{55}	+	17 920	$0, 2, 5, 7$	2
		ψ_6 ψ_7	$\circ$	16	χ_{54}	+	17 920	$0, 2, 5, 7$	2
	$3_+^{1+6}:(2.A_4 \times A_4).2$	2_5 2_6	$\circ$	2	χ_{41}	+	7 280	$0, 5, 7, 13$	2
		6_3 6_4	$\circ$	6	χ_{57}	+	21 840	$0, 5, 7, 13$	2
$2.\Omega_7(3)$	$3^5:(2.\mathrm{SU}_4(2):2)$	ψ'_{23} ψ''_{23}	$\circ$	20	χ_{69}	+	7 280	$0, 5, 7, 13$	2
		ψ'_{30} ψ''_{30}	$\circ$	60	χ_{87}	+	21 840	$0, 5, 7, 13$	2
		ψ'_{33} ψ''_{33}	$\circ$	64	χ_{88}	+	23 296	$0, 7, 13$	2
	$3^{3+3}:(2 \times \mathrm{SL}_3(3))$	$\zeta_2 \boxtimes \psi_4$ $\zeta_2 \boxtimes \psi_5$	$\circ$	16	χ_{85}	+	17 920	$0, 2, 5, 7$	2
		$\zeta_2 \boxtimes \psi_6$ $\zeta_2 \boxtimes \psi_7$	$\circ$	16	χ_{84}	+	17 920	$0, 2, 5, 7$	2
	$3_+^{1+6}:2.(2.A_4 \times A_4).2$	2_7 2_8	$\circ$	2	χ_{70}	+	7 280	$0, 5, 7, 13$	2
		6_5 6_6	$\circ$	6	χ_{86}	+	21 840	$0, 5, 7, 13$	2

Table 8. Parabolic examples (continued)

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Primes	Nts
$\mathrm{PSU}_6(2)$	$2^9:\mathrm{PSL}_3(4)$	ψ_6 ψ_7	$\circ$	45	χ_{46}	$+$	40 095	$0,3,5,11$	2
	$2^{4+8}:(\mathrm{SL}_2(2)\times\mathrm{SL}_2(4))$	$1\boxtimes\psi_2$ $1\boxtimes\psi_3$	$+$	3	χ_{38}	$+$	18 711	$0,3,7,11$	2
		$\zeta_2\boxtimes\psi_2$ $\zeta_2\boxtimes\psi_3$	$+$	3	χ_{37}	$+$	18 711	$0,3,7,11$	2
		$2\boxtimes\psi_2$ $2\boxtimes\psi_3$	$+$	6	χ_{45}	$+$	37 422	$0,7,11$	2
$3.\mathrm{PSU}_6(2)$	$2^9:\mathrm{SL}_3(4)$	ψ_{36} ψ_{37}	$\circ 2$	45	χ_{120}	$\circ 2$	40 095	$0,5,11$	2
	$2^{4+8}:(3\times\mathrm{SL}_2(2)\times\mathrm{SL}_2(4))$	$\zeta_3\boxtimes 1\boxtimes\psi_2$ $\zeta_3\boxtimes 1\boxtimes\psi_3$	$\circ 2$	3	χ_{112}	$\circ 2$	18 711	$0,7,11$	2
		$\zeta_3\boxtimes\zeta_2\boxtimes\psi_2$ $\zeta_3\boxtimes\zeta_2\boxtimes\psi_3$	$\circ 2$	3	χ_{111}	$\circ 2$	18 711	$0,7,11$	2
		$\zeta_3\boxtimes 2\boxtimes\psi_2$ $\zeta_3\boxtimes 2\boxtimes\psi_3$	$\circ 2$	6	χ_{119}	$\circ 2$	37 422	$0,7,11$	2
$F_4(2)$	$[2^{20}]:(\mathrm{SL}_2(2)\times\mathrm{SL}_3(2))$	$\zeta_2\boxtimes\psi_2$ $\zeta_2\boxtimes\psi_3$	$\circ$	3	χ_{82}	$+$	9 398 025	$0,3,5,13,17$	2

Table 9. Parabolic examples (continued), $\mathrm{char}(K) = 0$

G	H	ψ	ind	$\psi(1)$	χ	ind	$\chi(1)$	Nts
${}^2E_6(2)$	$2^{8+16}:\Omega_8^-(2)$	ψ_{16} ψ_{17}	$+$	1 071	χ_{94}	$+$	24 748 759 035	2
		ψ_{23} ψ_{24}	$+$	2 142	χ_{115}	$+$	49 497 518 070	2
		ψ_{29} ψ_{30}	$+$	2 835	χ_{121}	$+$	65 511 420 975	2
		ψ_{31} ψ_{32}	$+$	2 835	χ_{122}	$+$	65 511 420 975	2
	$[2^{29}]:(\mathrm{SL}_2(2) \times \mathrm{PSL}_3(4))$	$\zeta_2 \boxtimes \psi_6$ $\zeta_2 \boxtimes \psi_7$	$\circ$	45	χ_{116}	$+$	53 033 055 075	
	$[2^{31}]:(\mathrm{SL}_2(4) \times \mathrm{SL}_3(2))$	$\psi_2 \boxtimes \psi_2$ $\psi_3 \boxtimes \psi_3$	$\circ$	9	χ_{103}	$\circ$	31 819 833 045	2
		$\psi_2 \boxtimes \psi_3$ $\psi_3 \boxtimes \psi_2$	$\circ$	9	χ_{104}	$\circ$	31 819 833 045	2
$3.{}^2E_6(2)$	$2^{8+16}:(3 \times \Omega_8^-(2))$	$\zeta_3 \boxtimes \psi_{16}$ $\zeta_3 \boxtimes \psi_{17}$	$\circ 2$	1 071	χ_{287}	$\circ 2$	24 748 759 035	2
		$\zeta_3 \boxtimes \psi_{23}$ $\zeta_3 \boxtimes \psi_{24}$	$\circ 2$	2 142	χ_{323}	$\circ 2$	49 497 518 070	2
		$\zeta_3 \boxtimes \psi_{29}$ $\zeta_3 \boxtimes \psi_{30}$	$\circ 2$	2 835	χ_{335}	$\circ 2$	65 511 420 975	2
		$\zeta_3 \boxtimes \psi_{31}$ $\zeta_3 \boxtimes \psi_{32}$	$\circ 2$	2 835	χ_{336}	$\circ 2$	65 511 420 975	2
	$[2^{29}]:(\mathrm{SL}_2(2) \times \mathrm{SL}_3(4))$	$\zeta_2 \boxtimes \psi_{36}$ $\zeta_2 \boxtimes \psi_{37}$	$\circ 2$	45	χ_{325}	$\circ 2$	53 033 055 075	
	$[2^{31}]:(3 \times \mathrm{SL}_2(4) \times \mathrm{SL}_3(2))$	$\zeta_3 \boxtimes \psi_2 \boxtimes \psi_2$ $\zeta_3 \boxtimes \psi_3 \boxtimes \psi_3$	$\circ 2$	9	χ_{299}	$\circ 2$	31 819 833 045	2
		$\zeta_3 \boxtimes \psi_2 \boxtimes \psi_3$ $\zeta_3 \boxtimes \psi_3 \boxtimes \psi_2$	$\circ 2$	9	χ_{300}	$\circ 2$	31 819 833 045	2

5.2. The proofs

The proof of Theorem 5.2 follows the same line as the one of Theorem 3.2 for the sporadic groups. We will thus be rather sketchy here. Let G be quasisimple group such that $G/Z(G)$ is one of the simple groups of Tables 1 or 2. With a few exceptions, the ordinary and modular character tables of G and all of its relevant maximal subgroups are available in GAP. Many of the character tables of the maximal subgroups have been computed by Sebastian Dany in his Diploma thesis [**24**]. For the construction of some of the subgroups involved, Rob Wilson's Atlas of Group Representations [**109**] was used. With all this information at hand, the proof is tedious but straightforward. For the results in the case of $\mathrm{char}(K) = \ell > 0$, Lemma 2.11 is invoked most of the time. If G is such that $G/Z(G)$ is an alternating group, i.e., isomorphic to one of A_5, A_6 or A_8, some of our results are also contained in [**30**] and [**90**], and, of course, in Chapter 4.

In the following we comment on those groups G where some additional arguments were needed. This is the case if either some modular character table of G is not (completely) available or some character tables of relevant subgroups of G are missing.

As before, if H is a maximal subgroup of G, by ψ we denote a K-character of H such that $\chi := \mathrm{Ind}_H^G(\psi)$ is irrreducible.

5.2.1. Induction from non-parabolic subgroups. We begin with the cases where $H/Z(G)$ is a non-parabolic subgroup of $G/Z(G)$.

$G = 12_1.\mathrm{PSU}_4(3)$ or $12_2.\mathrm{PSU}_4(3)$: Let H be a maximal subgroup of G such that some K-character of H induces to an irreducible K-character of G. Using the known (modular) character tables of G, one easily checks that this implies $[G\colon H] = 126$, and thus $H = 12.\mathrm{SU}_4(2)$. (For a more precise description of the structure of H see Remark 5.3.) We also find that every modular character of H which could possibly induce to an irreducible character of G must be liftable. In view of Lemma 2.11 it suffices to look at the ordinary characters of H. Since these are available in GAP, the results given in Table 4 are readily obtained.

$G = 2.G_2(4)$: There are three non-parabolic subgroups of $G/Z(G)$ whose index is small enough to be possible block stabilizers. These are $2.J_2$, $2.(\mathrm{SU}_3(4)\colon 2)$, and $2.(3.\mathrm{PSL}_3(4)\colon 2_3)$. If $H = 2.(3.\mathrm{PSL}_3(4)\colon 2_3)$, then $[G\colon H] = 2\,080$, and there is exactly one ordinary irreducible character of G, whose degree is a multiple of this index. This degree equals $2{\cdot}2\,080$, but H does not have an irreducible character of degree 2. The only candidate degree in characteristics $\ell > 0$ occurs for $\ell = 13$,

and is the same as above. Since 13 does not divide $|H|$, this subgroup does not give rise to an imprimitive irreducible character of G.

Next, let $H = 2.(\mathrm{SU}_3(4){:}\,2)$, where $[G{:}\,H] = 2016$. The only multiple of 2 016 which is equal to the degree of an irreducible K-character of G is 2 016, and exactly one such character exists for $\ell = 0, 3, 7$. The subgroup H is isoclinic to but not isomorphic to $2 \times (\mathrm{SU}_3(4){:}\,2)$. (GAP does not find a possible fusion of conjugacy classes of the group $2 \times (\mathrm{SU}_3(4){:}\,2)$ into G.) Let K have one of the above characteristics. There are two K-characters of H of degree 1, arising from dual representations, inducing to the irreducible character of G of degree 2 016.

Finally, let $H = 2.J_2$. The only possible degree of a K-character inducing to an irreducible character of G is 14. This can only happen if $\ell = 0, 7, 13$. One checks with GAP that the two characters of degree 14 induce to reducible characters of G if $\ell = 0$. This also rules out the cases $\ell = 7, 13$.

$G = 2.F_4(2)$: According to the "Improvements to the Atlas" in [**65**, 297–327], the list of maximal subgroups of $2.F_4(2)$ given in [**21**, p. 170] is complete. There are only three non-parabolic maximal subgroups H of G with $|H|^2 \geq |G|$, namely $2 \times \mathrm{Sp}_8(2)$, $2.(\Omega_8^+(2){:}\,S_3)$ and $2.({}^3D_4(2){:}\,3)$. Let H be one of these groups.

By the result of Seitz cited above (see Theorem 1.2), we may assume that $\mathrm{char}(K) \neq 2$. Suppose in addition that $\mathrm{char}(K) \neq 3$. Using the ordinary character table and the known (modular) character tables of G (see [**47**]), we check that no irreducible K-character of G has degree divisible by the index $[G{:}\,H]$, except possible if $H = 2.\mathrm{Sp}_8(2)$. In this case, however, the modular character tables of H are also known. It is then routine to rule out this possibility as well.

Finally, let char $(K) = 3$. If $H = 2.({}^3D_4(2){:}\,3)$ or $2.(\Omega_8^+(2){:}\,S_3)$, a K-character ψ of H inducing irreducibly to G has degree at most 5. It follows (as in the proof of Theorem 3.2 in the case $G = 2.\mathrm{Co}_1$ and $H = 2.(\mathrm{PSU}_6(2){:}\,S_3)$) that ψ is liftable. Since there are no examples in characteristic 0, we are done for these two subgroups by Lemma 2.11. It remains to consider the subgroup $H = 2 \times \mathrm{Sp}_8(2)$. Here, the only candidates for the character ψ have degrees 1, 35, 50, 118, 135, or 203. We rule out these possibilities using the decomposition matrix of H modulo 3 and Lemma 2.12.

$G := 6.{}^2E_6(2)$: According to the "Improvements to the Atlas" in [**65**, 297–327], the list of maximal subgroups of ${}^2E_6(2)$ given in [**21**, p. 191] is complete (up to a correction of the structure of one of the maximal subgroups which does not affect its order). If H is a maximal subgroup of G such that $H/Z(G)$ is not a parabolic subgroup of ${}^2E_6(2)$, then $H/Z(G)$ is one of $F_4(2)$, Fi_{22} or $\Omega_{10}^-(2)$. By a result of Seitz

(see Theorem 1.2), we may and will assume that $\mathrm{char}(K) \neq 2$ in the following. We will also make use of the ordinary character table of G computed by Frank Lübeck (see [**81**]).

Suppose first that $H/Z(H) = F_4(2)$. Then H is one of $6 \times F_4(2)$ or $3 \times 2.F_4(2)$ and $\psi(1) \leq 4\,478$. All irreducible K-characters of degrees smaller than or equal to $4\,478$ are known (see [**108, 47**]). In case of $\mathrm{char}(K) \neq 0$, all of these characters are liftable, except for two characters of degree $1\,104$ for $\mathrm{char}(K) = 7$, and one character of degree $1\,366$ for $\mathrm{char}(K) = 13$. Using GAP, we find that no ordinary irreducible character of H induces to an irreducible character of G. This also rules out the cases of the liftable characters by Lemma 2.11. The non-liftable cases are ruled out using Lemma 2.12. Suppose next that $H/Z(H) = \mathrm{Fi}_{22}$. Then $H = 6.\mathrm{Fi}_{22}$, the full covering group of Fi_{22} or $H = 2 \times 3.\mathrm{Fi}_{22}$ (GAP shows that there is no possible class fusion of $2.\mathrm{Fi}_{22}$ or Fi_{22} into G). Moreover, $\psi(1) \leq 87$. This implies that $\psi(1) \in \{77, 78\}$ (see [**50, 54, 91**]), and thus that the elements of order 3 of $Z(G)$ lie in the kernel of ψ. This case is now easily ruled out using Lemmas 2.11 and 2.12. Suppose finally that $H/Z(G) = \Omega_{10}^-(2)$. Then $\psi(1) \leq 33$. Since $\Omega_{10}^-(2)$ has a trivial Schur multiplier, we have $H = 6 \times \Omega_{10}^-(2)$. By [**71**], every irreducible K-character of H has degree larger than 33 if $\mathrm{char}(K) \neq 2$.

5.2.2. Induction from parabolic subgroups. Here we consider the cases where $H/Z(H)$ is a parabolic subgroup of $G/Z(G)$.

$G = 2.({}^2B_2(8))$: Every irreducible K-character of G whose degree is divisible by 65 has $Z(G)$ in its kernel. Thus it suffices to consider the simple group $G/Z(G)$, and its maximal parabolic subgroup. The claimed result follows from Harish-Chandra theory (or with GAP).

$G = 2.\mathrm{Sp}_6(2)$: The same arguments as above give the desired results.

$G = 12_1.\mathrm{PSU}_4(3)$ or $12_2.\mathrm{PSU}_4(3)$: There are two maximal subgroups H of G such that $H/Z(G)$ is a parabolic subgroup of $G/Z(G)$. Let us first consider the subgroup H of index 112 in G, i.e., $H/Z(G) \cong 3^4\!:\!\mathrm{PSL}_2(9)$. There are no faithful irreducible K-characters of G whose degree is divisible by 112. The analogous statement holds for the group $3_2.\mathrm{PSU}_4(3)$. The faithful irreducible characters of $3_1.\mathrm{PSU}_4(3)$ of degrees divisible by 112 have degree $3 \cdot 112 = 336$ and exist exactly in characteristics $0, 5$ and 7. It turns out that the inverse image of $3^4\!:\!\mathrm{PSL}_2(9)$ in $3_1.\mathrm{PSU}_4(3)$ does not have irreducible characters of degree 3 in these characteristics. Now consider the central quotient $4.\mathrm{PSU}_4(3) = \mathrm{SU}_4(3)$ of G. This is a finite group of Lie type, and thus we only have to consider Harish-Chandra induced representations from

the parabolic subgroup $3^4 : 2.(2 \times \mathrm{PSL}_2(9))$ (see Proposition 7.1). It is not difficult to construct the character table of this subgroup and check that only the examples given in Table 6 exist.

Next, let H be the maximal subgroup of index 280, so that $H/Z(G) \cong 3_+^{1+4} : 2S_4$. Since H is solvable, every modular irreducible character of H is liftable. By Lemma 2.11, it suffices to look at the ordinary characters of H. The groups $12_2.\mathrm{PSU}_4(3)$, $3_2.\mathrm{PSU}_4(3)$ and $3_1.\mathrm{PSU}_4(3)$ do not have any faithful irreducible characters of degrees divisible by 280. For $G = 12_1.\mathrm{PSU}_4(3)$, there are such characters for $\ell \in \{0, 5, 7\}$. All of these have degree $3 \cdot 280 = 840$. There are faithful irreducible characters of H of degree 3, but all of these induce to reducible characters of G. The same arguments as for the first parabolic subgroup of $G/Z(G)$ now give the desired results.

$G = 3.G_2(3)$: There are two conjugacy classes of maximal parabolic subgroups of $G/Z(G) = G_2(3)$, conjugate by an outer automorphism of $G/Z(G)$. The index of these subgroups is 364. There is no faithful irreducible K-character of G whose degree is divisible by 364. Thus it suffices to consider the simple group $G/Z(G)$, and any one of its maximal parabolic subgroups. The only possible degree for an induced irreducible K-character of $G/Z(G)$ is 728, and this can only occur for $\mathrm{char}(K) \in \{0, 7, 13\}$. Using GAP, it is easy to find the example given in Table 6.

$G = 2.\Omega_8^+(2)$:First, let $H \leq G$ have index 135. The possible degrees for ψ are 21, 24 and 45. If $\psi(1) = 21$ or 45, then $\mathrm{Ind}_H^G(\psi)(1) = 2\,835$ or $6\,075$. Any irreducible K-character of G of one of these degrees has $Z(G)$ in its kernel, and we may thus look at the simple Chevalley group $\Omega_8^+(2)$. In this case we easily find the examples given in Table 6. The possibility $\psi(1) = 24$ only occurs if $\mathrm{char}(K) = 0$ or 3. However, there is no ordinary irreducible character of H of degree 24. Using MOC (see [**48**]), we can show that there is no irreducible 3-modular character of degree 24 of H. Next, let $H \leq G$ have index $1\,575$. The only possible degree for χ turns out to be $1\,575$. Then χ has $Z(G)$ in its kernel and we may consider the parabolic subgroup $2_+^{1+8} : (S_3 \times S_3 \times S_3)$ of $\Omega_8^+(2)$. Using GAP, one shows that no character of degree 1 of this parabolic subgroup induces to an irreducible character of G.

$G = 2.G_2(4)$: There is no faithful irreducible K-character of G whose degree is divisible by the index of a maximal parabolic subgroup of $G/Z(G)$. Again, it suffices to consider the simple group $G/Z(G) = G_2(4)$. Using the known (modular) character tables and GAP, one easily obtains the result.

$G = 2.\Omega_7(3)$: Let H be a maximal subgroup of $6.\Omega_7(3)$ such that $H/Z(H)$ is a parabolic subgroup of $\Omega_7(3)$. The ordinary and modular character tables of $6.\Omega_7(3)$ are known and available in GAP. We use these to check that, with a few possible exceptions, no faithful irreducible character of $6.\Omega_7(3)$ or of $3.\Omega_7(3)$ has a degree divisible by the index of H in $6.\Omega_7(3)$. The exceptions occur for the largest parabolic subgroup $H = 6.(3^5 : (\mathrm{SU}_4(2) : 2))$ of index 364 and only if $\mathrm{char}(K) \neq 2, 3$. Let $\mathrm{char}(K) \in 0, 5$; then a possible faithful character of H inducing to an irreducible character of $6.\Omega_7(3)$ has degree 54. But H does not have any faithful irreducible K-character of this degree. (The ordinary character table of H is available in GAP, and one checks that every ordinary irreducible faithful character of H has degree divisible by 108.) Next, let H denote the maximal subgroup of index 364 of $3.O_7(3)$. In this case, a faithful character of H inducing to an irreducible character of $3.\Omega_7(3)$ has degree 27 and can only occur for $\mathrm{char}(K) = 0$. However, the ordinary irreducible characters of H of degree 27 induce to reducible characters of $3.\Omega_7(3)$.

We are thus left with the group of Lie type $G = 2.\Omega_7(3)$. The character tables of all maximal subgroups of G are contained in GAP. It is then routine to obtain the results of Table 7. Alternatively, one can use Harish-Chandra theory.

$G = 6.\mathrm{PSU}_6(2)$: All ordinary and modular character tables of G are available in GAP. From this it follows that the central elements of order 2 of G are in the kernel of every irreducible K-character of G which is induced from a character of a subgroup H such that $H/Z(G)$ is parabolic in $\mathrm{PSU}_6(2)$. Thus it suffcies to investigate the group $3.\mathrm{PSU}_6(2) = \mathrm{SU}_6(2)$, which is a finite group of Lie type. By Proposition 7.1 we may restrict to Harish-Chandra induced characters. Using in addition the ordinary character tables of the maximal parabolic subgroups of $\mathrm{SU}_6(2)$ as well as the modular character tables of their Levi subgroups, we obtain the results from Table 8.

$G = 2.F_4(2)$: Let us first consider a maximal (parabolic) subgroup H of index 69 615. Then $H/Z(G)$ is the centralizer in $F_4(2)$ of an involution $z \in F_4(2)$ in class $2A$ or $2B$. A class multiplication coefficients computation shows that there is a conjugate y of z such that zy lies in class $3A$, respectively $3B$, of $F_4(2)$. Hence there is an element $t \in F_4(2)$ of order 3 such that $t \in \langle z, z^t \rangle$. Lemma 2.3 and Corollary 2.7 imply that neither $H/Z(G)$ nor H are block stabilizers of imprimitive irreducible modules of $G/Z(G)$, respectively G. Next we consider a maximal subgroup H with $H/O_2(H) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_3(2)$. There are no faithful irreducible characters of G whose degree is a multiple of $[G : H]$. (Although the 3-modular character table of G is not completely known, the

results of [**47**, Theorem 2.2] are sufficient to prove the above claim even if $\mathrm{char}(K) = 3$.) Thus we may consider the Chevalley group $G/Z(G)$ and its parabolic subgroup $H/Z(H)$. By Theorem 1.2, we may assume that $\mathrm{char}(K) \neq 2$. No 7-modular irreducible character of $G/Z(G)$ has degree divisible by $[G\colon H]$. In the remaining characteristics, the only possible degree for ψ is 3. (Once more, in characteristic 3 the incomplete results of [**47**] suffice to see this.) The results of Table 8 now follow from Proposition 7.1 and Harish-Chandra theory.

$G := 6.{}^2E_6(2)$: Here, we assume that $\mathrm{char}(K) = 0$. The ordinary character table of G has been computed by Frank Lübeck (see [**81**]). Let H be a maximal subgroup of G such that $H/Z(G)$ is one of the four maximal parabolic subgroups of $G/Z(G) = {}^2E_6(2)$. As a first step we show that the central element of order 2 of G is in the kernel of every K-character of H inducing to an irreducible K-character of G. First, let H be the largest of these four subgroups, so that $H/Z(H) = 2^{1+20}\colon \mathrm{PSU}_6(2)$. The image $\bar{H}$ of H in the factor group $\bar{G} := 3.{}^2E_6(2)$ is equal to the centralizer of an involution z. Using class multiplication coefficients one can show that that z together with some conjugate generates a dihedral group of order 6. Thus $\bar{H}$ and H cannot be block stabilizers in $\bar{G}$, respectively G by Corollary 2.7 and Lemma 2.3. Now let H be one of the remaining three maximal subgroups considered here. Then there is no faithful irreducible K-character of G whose degree is divisible by the index $[G\colon H]$. The analogous statement holds for the factor group $\tilde{G} := 2.{}^2E_6(2)$ with one exception. This occurs for the second maximal subgroup $\tilde{H}$ of $\tilde{G}$, where $\tilde{H}/Z(\tilde{G}) = 2^{8+16}\colon \Omega_8^-(2)$. In this case, the only possible degree for ψ is 4 480. Using Clifford theory it is not hard to show that $\tilde{H} = 2.(2^{8+16}\colon \Omega_8^-(2)) = 2.(2^{8+16})\colon \Omega_8^-(2))$ does not have an irreducible character of this degree.

It suffices now to consider the group $\bar{G} = 3.{}^2E_6(2)$. Note that $\bar{G}$ is a finite group of Lie type. Let $\bar{H}$ denote the image of H in $\bar{G}$. By Proposition 7.1, the unipotent radical of $\bar{H}$ is in the kernel of ψ, and we may thus use Harish-Chandra theory to obtain the results of Table 6. We omit the details (but see Chapters 7, 8, 10).

CHAPTER 6

Groups of Lie type: Induction from non-parabolic subgroups

In this chapter G is a quasisimple covering group of a simple group of Lie type. By the latter we understand any finite simple group which is not isomorphic to a sporadic simple group nor to an alternating group A_7 or A_n, $n \geq 9$. Throughout this chapter we also assume that $G/Z(G)$ does not have an exceptional multiplier, and that $G/Z(G)$ is not isomorphic to a simple group of Lie type of a different defining characteristic. The cases excluded here were treated in Chapter 5. Moreover, K is an algebraically closed field of characteristic $\ell \geq 0$, but not equal to the defining characteristic of G. In Proposition 5.1 we have already observed that the Tits simple group ${}^2F_4(2)'$ does not have any imprimitive irreducible module over K. For this reason we also exclude the Tits group from our considerations here. In particular, G is a finite group of Lie type. The main result of this chapter is the following reduction theorem.

THEOREM 6.1. *Let G be a quasisimple covering group of a simple group of Lie type subject to the restrictions formulated above, and let H be a maximal subgroup of G and a block stabilizer of an imprimitive irreducible KG-module. Then H is a parabolic subgroup of G.*

The proof of this theorem is given in Sections 6.2 and 6.3.

6.1. Outline of the strategy

Our starting point is the observation that the index of a block stabilizer is at most equal to the dimension of an absolutely irreducible KG-module. Convenient upper bounds for the latter are due to Seitz.

LEMMA 6.2 (Seitz). *Let M be an absolutely irreducible KG-module and B a Borel subgroup of G. Then* $\dim(M) \leq [G:B]c_G$ *where either* $c_G = 1$ *or is as given below.*

$$c_G = \begin{cases} (q^2-1)/(q^2-q+1), & \text{if } G \cong \mathrm{SU}_n(q), \quad n \text{ odd},\\ (q-1)/(q-\sqrt{2q}+1), & \text{if } G \cong {}^2B_2(q), \quad q = 2^{2m+1},\\ (q-1)/(q-\sqrt{3q}+1), & \text{if } G \cong {}^2G_2(q), \quad q = 3^{2m+1},\\ (q-1)^2/(q-\sqrt{2q}+1)^2, & \text{if } G \cong {}^2F_4(q), \quad q = 2^{2m+1}.\end{cases}$$

PROOF. This is due to Seitz [**98**, Theorem 2.2]. $\diamond$

In order to prove Theorem 6.1, we may assume that H is a maximal subgroup of G. In a first step we determine those non-parabolic maximal subgroups H of G for which $[G : H] > [G : B]c_G$. Such an H cannot be the block stabilizer of an imprimitive irreducible KG-module. When G is a classical group, we treat the various non-parabolic subgroups H according to their Aschbacher classes. Given G and H, we usually produce a lower bound b for the order of a Borel subgroup B of G, and a strict upper bound h for the order of H. Then $|H|c_G/|B| < hc_G/b$, and thus $hc_G/b \leq 1$ implies $[G : H] > [G : B]c_G$. In the Tables 1–6, the columns headed bc_G^{-1} and h give lower and upper bounds for $|B|c_G^{-1}$ and $|H|$, respectively. The column headed $\tilde{H}$ describes groups closely related to H, usually overgroups of H (in overgroups of G); in any case, $|H| \leq |\tilde{H}|$, and h is an upper bound for $|\tilde{H}|$. These are derived from the following easily obtained upper bounds for the orders of certain classical groups:

$$\begin{aligned}
|\mathrm{GL}_n(q)| &\leq q^{n^2};\\
|\mathrm{GU}_n(q)| &\leq (q+1)q^{n^2-1};\\
|\mathrm{Sp}_n(q)| &\leq q^{n(n+1)/2};\\
|\mathrm{GO}_n^{\epsilon}(q)| &\leq q^{n(n-1)/2}, \quad \epsilon \in \{0,+,-\}.
\end{aligned}$$

For those H surviving the first step, we attempt to construct an element $t \in G \setminus H$ such that t centralizes $H \cap H^t$, in which case Lemma 2.2 rules out H as possible block stabilizer. At this point very few possibilities for G and H remain. These are disposed of by character theoretic methods similar to those from Chapters 3 and 5. Here, Frank Lübeck's data base [**80**] containing the ordinary character degrees of groups of Lie type of small rank is particularly helpful.

6.2. The classical groups of Lie type

For further reference, we collect the groups considered here in the following hypothesis.

HYPOTHESIS 6.3. Throughout this section G denotes one of the following quasisimple classical groups.

(a) $\mathrm{SL}_n(q)$ with $n = 2$ and $q \geq 11$ or $n \geq 3$ and $(n,q) \neq (3,2)$, $(3,4)$, $(4,2)$;
(b) $\mathrm{SU}_n(q)$ with $n \geq 3$, and $(n,q) \neq (3,2)$, $(3,3)$, $(4,2)$, $(4,3)$, $(6,2)$;
(c) $\mathrm{Sp}_n(q)$ with $n \geq 4$ even, and $(n,q) \neq (4,2)$, $(4,3)$, $(6,2)$;
(d) $\Omega_n(q)$ with q odd and $n \geq 7$ odd and $(n,q) \neq (7,3)$;

(e) $\Omega_n^+(q)$ with $n \geq 8$ even and $(n,q) \neq (8,2)$;
(f) $\Omega_n^-(q)$ with $n \geq 8$ even.

The spin groups, which are excluded from our list, are treated as follows. Suppose that $G = \Omega_n^\epsilon(q)$ is one of the orthogonal groups, and that H is a non-parabolic subgroup of G. If the conclusion of Theorem 6.1 for G and H is due to the fact that $[G:H] > [G:B]$ (notice that $c_G = 1$ in this case), then the inverse image of H in $\mathrm{Spin}_n^\epsilon(q)$ is not a block stabilizer either. If $\mathrm{Spin}_n^\epsilon(q) \neq \Omega_n^\epsilon(q)$ and we don't prove that $[G:H] > [G:B]$, we find an element $t \in G$ of odd order such that t centralizes $H \cap H^t$. Then the statement of Theorem 6.1 for the inverse image of H in $\mathrm{Spin}_n^\epsilon(q)$ follows from Corollary 2.7.

The natural module for G is denoted by V. Thus, if G is the special unitary group $\mathrm{SU}_n(q)$, then $V = \mathbb{F}_{q^2}^n$, and $V = \mathbb{F}_q^n$ if G is one of the other groups. Notice that by our restrictions on (n,q), each of the classical groups considered above has a unique natural module.

Recall from the introduction that the maximal subgroups of G are grouped into Aschbacher classes. We use the definitions of these classes established in Kleidman-Liebeck's book [**69**]. In the sequel, we consider each Aschbacher class in turn.

6.2.1. The case H is of type $\mathcal{C}_1$ but not parabolic. In this subsection either G is one of the groups defined in Hypothesis 6.3 or $G = \Omega_n(q)$ with $n \geq 5$, where q can be any prime power. We define a $\mathcal{C}_1^*$-type subgroup of G to be a non-parabolic subgroup of type $\mathcal{C}_1$. We wish to deal with a certain $\mathcal{C}_7$-type subgroup of $G = \mathrm{Sp}_4(q)$ as a $\mathcal{C}_1^*$-type subgroup of $\Omega_5(q) \cong G$ (see Lemma 6.20), and with certain $\mathcal{C}_8$-type subgroups of $G = \mathrm{Sp}_n(q)$, q even, as $\mathcal{C}_1^*$-type subgroups of $\Omega_{n+1}(q) \cong G$ (see 6.2.7). This is the reason for enlarging the collection of groups considered in this subsection. We note that if G is linear, then it does not have any $\mathcal{C}_1^*$-type subgroups.

We begin by describing the configurations we have to consider. Although the result is well known, we sketch its proof for the convenience of the reader.

LEMMA 6.4. *Let H be a maximal $\mathcal{C}_1^*$-type subgroup of G. Then one of the following occurs.*

(1) *We have $G = \Omega_n^\epsilon(q)$ with $n \geq 8$ even and q even, and H is the stabilizer of a hyperplane Y of V with $\dim(Y \cap Y^\perp) = 1$. In particular, $H = \Omega_{n-1}(q) \cong \mathrm{Sp}_{n-2}(q)$.*

(2) *We have $G = \Omega_n(q)$ with $n \geq 5$ odd and q even, and either H is isomorphic to a maximal $\mathcal{C}_1^*$-type subgroup of $\mathrm{Sp}_{n-1}(q)$ through the isomorphism $G \to \mathrm{Sp}_{n-1}(q)$, or H is the stabilizer of a non-degenerate*

hyperplane Y of V. In the latter case, $H = \mathrm{GO}_{n-1}^{\epsilon}(q)$, where ϵ depends on the Witt index of Y.

(3) *Otherwise H stabilizes a pair of complementary, mutually orthogonal, non-isometric, non-degenerate subspaces X and Y of V. If q is odd, we have $Y = X^{\perp}$.*

PROOF. If G is orthogonal, let Q denote the quadratic form on V defining G. Suppose that H stabilizes the proper subspace $0 \neq W \leq V$. Then it also stabilizes $W^{\perp}$ and $\mathrm{rad}(W) := W \cap W^{\perp}$ (if G is orthogonal, $\mathrm{rad}(W)$ is defined with respect to the polar form of Q). If $\mathrm{rad}(W) = 0$, i.e., if W is non-degenerate, we are in Case (3).

Suppose first that q is odd or that G is not orthogonal. As H is maximal and not parabolic, it follows that W is non-degenerate.

Now suppose that q is even, that G is orthogonal and that $\mathrm{rad}(W) \neq 0$. Put

$$U := \{v \in \mathrm{rad}(W) \mid Q(v) = 0\}.$$

Then U is an H-invariant subspace of V. As H is maximal and non-parabolic, it follows that $U = 0$. This implies that $\mathrm{rad}(W)$ is 1-dimensional.

Now suppose that n is even. As H stabilizes $Y := \mathrm{rad}(W)^{\perp}$, we are in Case (1).

Finally suppose that n is odd. Then $\mathrm{rad}(V)$ is 1-dimensional. If $W + \mathrm{rad}(V) = V$, we are in Case (2). Thus suppose that $W + \mathrm{rad}(V) \neq V$. As $W + \mathrm{rad}(V)$ is H-invariant, we find that $\mathrm{rad}(V) \leq W$ and $\mathrm{rad}(V) = \mathrm{rad}(W)$. It follows that $\mathrm{rad}(V) \neq W$. Now let

$$\bar{\ }: V \to \bar{V} := V/\mathrm{rad}(V)$$

denote the canonical epimorphism. This gives rise to the isomorphism

$$\bar{\ }: \Omega(V) \to \mathrm{Sp}(\bar{V}).$$

As $\bar{H}$ is a non-parabolic subgroup of $\mathrm{Sp}(\bar{V})$, it follows that $\bar{W}$ is a non-zero, proper, non-degenerate subspace of $\bar{V}$. This completes the proof. $\diamond$

We first treat the case of odd defining characteristic.

LEMMA 6.5. *If q is odd and H is a maximal $\mathcal{C}_1^*$-type subgroup of G, there exists an element $t \in G \setminus H$ of odd order centralizing $H \cap H^t$.*

PROOF. Using the notation of Lemma 6.4, we choose $z \in \mathrm{GL}(V)$ so that it acts via $-I_X$ on X and via I_Y on Y. Evidently z is an involution and $H \leq C_G(z)$.

We first assume that $q \neq 3$ and pick non-zero vectors $x \in X$ and $y \in Y$. Thus with respect to the basis $\{x, y\}$, the element z acts diagonally on $U := \langle x, y\rangle$ with eigenvalues ± 1.

If G is symplectic, x and y are necessarily singular and perpendicular, as $Y = X^{\perp}$. Consequently U is totally singular. Choose $x' \in X$ and $y' \in Y$ so that (x, x') and (y, y') are hyperbolic pairs, and put $U' := \langle x', y' \rangle$. The $U \oplus U'$ is a non-degenerate subspace of V. As $z \in G$ we see that z lies in the Levi subgroup L_U of G stabilizing U and U'. Now L_U contains a z_U-invariant direct summand isomorphic to $\mathrm{GL}(U) \cong \mathrm{GL}_2(q)$. By Lemma 2.8, there are maximal tori T_1 and T_2 of $\mathrm{SL}(U)$ of orders $q-1$ and $q+1$, respectively, inverted by z_U. As $q > 3$, there is a non-trivial element $t_0 \in T_1 \cup T_2$ of odd order. Let $t \in G$ be the element which acts as t_0 on U and as the identity on $(U \oplus U')^{\perp}$. Then z inverts t and hence $t \in C_G(H \cap H^t)$ by Lemma 2.3.

If G is unitary or orthogonal we pick x and y so that they are nonsingular with respect to the form defining G. Thus, as x and y are perpendicular, U is non-degenerate with respect to the form defining G. Now if G is unitary, $\mathrm{SU}(U) \times \mathrm{SU}(U^{\perp}) \leq N_G(U)$ and $\mathrm{SU}(U)$ is z_U invariant. Using Lemma 2.8, an argument analogous to the one in the symplectic case shows that there exists a $t \in C_G(H \cap H^t)$ with t of odd order.

If G is orthogonal, $G = \Omega(V) \cong \Omega^{\epsilon}(q)$, we adjust the choice of y so that the order of $\mathrm{SO}(U)$ is twice an odd number. Since $q \neq 3$ this is always possible. Now $\mathrm{SO}(U) \times \mathrm{SO}(U^{\perp}) \leq \mathrm{SO}(V) \cong \mathrm{SO}^{\epsilon}(q)$. Since $z_U \in \mathrm{GO}(U) \setminus \mathrm{SO}(U)$, we have $\langle z_U, \mathrm{SO}(U) \rangle = \mathrm{GO}(U)$; the latter is a dihedral group. Let $t \in \mathrm{SO}(U)$ be a nontrivial element of odd order, viewed as an element of $\mathrm{SO}(V)$, by letting it act as the identity on $U^{\perp}$. As t has odd order, it lies in $\Omega(V) = G$. Since z inverts t, our claim follows from Lemma 2.3.

We now consider the case where $q = 3$ and G is symplectic. The group $\mathrm{Sp}_4(3)$ contains a unique class of non-central involutions, and a subgroup isomorphic to S_3. We choose a non-degenerate subspace $U \leq V$ of dimension 4, whose intersections with X and Y are 2-dimensional. Then z_U is a non-central involution in $\mathrm{Sp}(U) \cong \mathrm{Sp}_4(3)$. Let $t \in G$ be an element of order 3 fixing U such that t_U is inverted by z and such that t acts as the identity on $U^{\perp}$. Then z inverts t and we are done by Lemma 2.3.

We now consider the case where $q = 3$ and $G = \mathrm{SU}_n(3)$, $n \geq 3$. The group $\mathrm{SU}_3(3)$ contains a unique class of involutions, and a subgroup isomorphic to S_3. We now choose a non-degenerate subspace $U \leq V$ of dimension 3, whose intersection with X is 2-dimensional. Then z_U is an involution in $\mathrm{SU}(U) \cong \mathrm{SU}_3(3)$. Let $t \in G$ be an element of order 3 fixing U such that t_U is inverted by z and such that t acts as the identity on $U^{\perp}$. Then z inverts t and we are done by Lemma 2.3.

We now consider the case where $q = 3$ and G is orthogonal. The group $\Omega_5(3)$ has two classes of involutions, distinguished by the their centralizer order. Let z_0 be an involution in $\Omega_5(3)$ whose centralizer has order $2^6 \cdot 3^2$. (The centralizer of an involution in the other class has order $2^5 \cdot 3$.) A computation of class multiplication coefficients shows that there is a conjugate z_0' of z_0 such that the product $z_0 z_0'$ has order 3. In particular, z_0 inverts an element of order 3. These facts can easily be checked using the known character table of $\Omega_5(3) \cong \mathrm{PSp}_4(3)$ (see [**37**]). We now choose a non-degenerate subspace $U \leq V$ of dimension 5, whose intersection with X is 4-dimensional. Now $z_U \in \Omega_5(3)$ since $-I_{U\cap X}$ is contained in $\Omega_4^+(3)$. Hence z_U is a non-central involution in $\Omega_5(3)$, and, by the above, inverts an element of order 3. Let $t \in G$ be an element of order 3 fixing U such that t_U is inverted by z and such that t acts as the identity on $U^\perp$. Then z inverts t and we are done by Lemma 2.3. $\diamond$

Next we deal with the groups of even characteristic, where we distinguish various cases.

Lemma 6.6. *Let $G = \Omega_n^\epsilon(q)$ with $n \geq 8$ even and q even, and suppose that H is the stabilizer of a hyperplane Y of V with* $\dim(Y \cap Y^\perp) = 1$. *Then there is $t \in G \setminus H$ such that $t \in C_G(H \cap H^t)$.*

Proof. There is an involution $z \in \mathrm{GO}_6^\epsilon(2) \leq \mathrm{GO}_n^\epsilon(q)$ with $H = C_G(z)$. Suppose first that $G = \Omega_n^+(q)$. Now $\mathrm{GO}_6^+(2) \cong S_8$ and there is $z' \in \mathrm{GO}_6^+(2)$ conjugate to z such that $t := zz' \in \Omega_6^+(2) \leq G$ has order 3. An analogous construction works for $\Omega_n^-(q)$ since $\mathrm{GO}_6^-(2) \cong \mathrm{SU}_4(2){:}2$. The claim follows from Lemma 2.3. $\diamond$

The following lemma deals with the generic case.

Lemma 6.7. *Let G be one of the groups $\mathrm{SU}_n(q)$, $n \geq 3$, q even, $\mathrm{Sp}_n(q)$, $n \geq 4$, q even, or $\Omega_n^\epsilon(q)$, $n \geq 8$ even and q even. Suppose that H is a maximal subgroup of $\mathcal{C}_1^*$-type, fixing a pair of non-degenerate, mutually orthogonal, complementary subspaces X and Y. Then there exists an element $t \in G \setminus H$ such that $t \in C_G(H \cap H^t)$.*

Proof. For any subspace $W \leq V$, write $W' := W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ and view $\mathrm{GL}(V)$ as a subgroup of $\mathrm{GL}(V')$. Let $\alpha \in \mathbb{F}_{q^2}^*$ be an element of order 3, and let $z \in \mathrm{GL}(V')$ be the element that acts as αI on X' and as $\alpha^2 I$ on Y'. Then $H = C_G(z)$.

First we assume that both X and Y are at least 2-dimensional, and pick 2-dimensional subspaces X_0 and Y_0 of X and Y, respectively such that $U := X_0 \oplus Y_0$ is a non-degenerate subspace of V. If G is orthogonal, we assume in addition that both X_0 and Y_0 are of plus-type. If G is orthogonal, we choose bases of X_0 and Y_0 consisting

of hyperbolic pairs. In the other cases we may choose bases of U so that the bilinear, respectively sesqui-linear form of G restricted to U is represented by J_4 (which, in the orthogonal case represents the polar form of the quadratic form restricted to U). We may also assume that $z_{U'}$ is represented by

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

Now let

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $t \in G$ and $t = [z, (z^t)^{-1}]$ and thus $t \in \langle z, z^t \rangle$. Our claim follows from Lemma 2.3.

Next assume that G is unitary and that X is 1-dimensional. As X is non-degenerate and $\dim(Y) \geq 2$, we can find a non-degenerate subspace Y_0 of Y of dimension 2, such that the Gram matrix of the form defining G, restricted to $U := X \oplus Y_0$ equals J_3 (with respect to a suitable basis of U). With respect to the same basis, we define $t_0 \in \mathrm{SU}(U)$ by the matrix

$$t_0 := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \alpha & 1 & 1 \end{pmatrix}.$$

Let $t \in G$ be such that $t_U = t_0$ and $t_{U^\perp} = I_{U^\perp}$. Again $t = [z, (z^t)^{-1}] \in \langle z, z^t \rangle$, so that our claim follows from Lemma 2.3.

Finally assume that G is orthogonal and that X is 2-dimensional and totally anisotropic, i.e. of minus-type. As $\dim(Y) \geq 6$, there is a non-degenerate 2-dimensional subspace $Y_0 \leq Y$ of plus-type. Put $U := X \oplus Y_0$. We identify X with $\mathbb{F}_{q^2}$, equipped with the norm form. Then $\Omega(X)$ equals the set of right multiplications by elements of norm 1. Let $x \in \mathbb{F}_{q^2}^*$ be an element of order $q+1$. Then multiplication by x is represented by the matrix

$$z_0 := \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$$

with respect to the $\mathbb{F}_q$-basis $\{1, x\}$ of $\mathbb{F}_{q^2}$. Choose a hyperbolic pair $\{y_1, y_2\}$ of Y_0, and an ordered basis of V of the form $\{y_1, x_1, x_2, y_2, \ldots\}$, where the dots indicate a basis of $U^\perp$, and where $x_1, x_2 \in X$ correspond

to $1, x$, respectively, in our identification of X with $\mathbb{F}_{q^2}$. Let $z \in G$ be the element represented by the matrix

$$z := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

with respect to this basis. Then $H = N_G(\langle z \rangle) = \langle C_G(z), s \rangle$, where $s \in G$ is an involution swapping the basis elements x_1, x_2 of X and leaving fixed y_1 and y_2. Now let t be the element

$$t := \begin{pmatrix} t_0 & 0 \\ 0 & I \end{pmatrix},$$

where t_0 is the matrix

$$t_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ a & 1 & 1 & 1 \end{pmatrix}$$

with respect to the basis $\{y_1, x_1, x_2, y_2\}$ of U. Then $t \in G$.

The fixed space of z and of z^t equals $\langle y_1, Y_1 \rangle$, where Y_1 denotes the orthogonal complement of Y_0 in Y. Hence $H \cap H^t = N_G(\langle z \rangle) \cap N_G(\langle z^t \rangle)$ stabilizes $\langle y_1, Y_1 \rangle$ and in turn also $\langle y_1, Y_1 \rangle^\perp = \langle y_1, X \rangle =: U_0$. Thus an element $g \in H \cap H^t$ has the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{6.1}$$

where A is a 4×4-matrix, B a $4 \times (n-4)$-matrix whose first three rows are zero, and C is an $(n-4) \times 4$-matrix whose last three columns are zero.

Now

$$g^t = \begin{pmatrix} t_0^{-1} A t_0 & t_0^{-1} B \\ C t_0 & D \end{pmatrix}.$$

The properties of C and D mentioned above imply $t_0^{-1} B = B$ and $C t_0 = C$. Our claim follows once we can show that t_0 commutes with the matrices A arising in (6.1).

We have already observed above that $H \cap H^t$ stabilizes U_0. It also stabilizes $X \cap Xt = \langle x_1 + x_2 \rangle$, and hence $x_1 + x_2$. As $H \cap H^t$ also fixes y_1, it follows that $(H \cap H^t)_{U_0} = \langle s_{U_0} \rangle$. As H stabilizes X and $Y = X^\perp$

the matrices A arising in (6.1) are of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

or

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

for suitable elements $b \in \mathbb{F}_q$. Such elements obviously commute with t_0. This completes our proof. $\diamond$

We finally consider the orthogonal groups of odd dimension and even characteristic.

LEMMA 6.8. *Let $G = \Omega_n(q)$ with $n \geq 5$ odd and q even, and suppose that H is the stabilizer of a non-degenerate hyperplane Y of V. Then there is $t \in G \setminus H$ such that t centralizes $H \cap H^t$.*

PROOF. Notice that Y is a complement to $X := V^{\perp}$. Choose a non-degenerate subspace Y_0 of Y of dimension 2 of plus-type, i.e., Y_0 contains isotropic vectors. Let Y_1 denote the orthogonal complement of Y_0 in Y, and put $U := X \oplus Y_0$.

Choose a hyperbolic pair $\{y, y'\}$ of Y_0 and a non-zero element $x \in X$. Let t_0 be the element of $\Omega(U)$ defined by the following matrix with respect to the basis $\{x, y', y\}$ of U:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Extend t_0 to all of V by letting it act as the identity on Y_1. In particular, t fixes y and x. Now $H \cap H^t$ stabilizes $Y \cap Yt$. As $y \in Y \cap Y^t$, we have $Y \cap Yt = Y_1 \oplus \langle y \rangle$, with $\langle y \rangle = \mathrm{rad}(Y \cap Yt)$. In particular, the elements of $H \cap H^t$ fix y and x.

Consider the basis $\{x, y', y, y_3, \ldots, y_n\}$ of V, where $\{y_3, \ldots, y_n\}$ is a basis of Y_1. Let $g \in H \cap H^t$, written with respect to this basis as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is a 3×3-matrix. Then

$$g^t = \begin{pmatrix} t_0^{-1} A t_0 & t_0^{-1} B \\ C t_0 & D \end{pmatrix}.$$

As g fixes y and x, the first and last row of B consist of zeroes, and thus $t_0^{-1}B = B$. As g stabilizes $\langle y \rangle \oplus Y_1$, the first two columns of C consist of zeroes, and thus $Ct_0 = C$.

Using the fact that g fixes x and y, and applying the invariance of the polar form to the pair (y, y'), we see that A is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

for some $\beta \in \mathbb{F}_q$. It follows that D commutes with t_0, proving our assertion. $\diamond$

We summarize the results of this subsection.

Proposition 6.9. *Let G be one of the following groups.*

(a) *A group as in Hypothesis 6.3.*

(b) *$G = \Omega_n(q)$, $n \geq 5$, q any prime power.*

(c) *$G = \mathrm{Spin}_n^{\epsilon}(q)$, $n \geq 5$, with q odd, $\epsilon \in \{-1, 0, 1\}$ and $(n, q) \neq (7, 3)$.*

Let H be a non-parabolic maximal subgroup of G of $\mathcal{C}_1$-type. Then H is not the stabilizer of an imprimitivity decomposition of an irreducible KG-module.

Proof. The possibilities for H are described in Lemma 6.4. The various cases where G is not a spin group are treated in Lemmas 6.5, 6.6, 6.7 and 6.8. If G is a spin group, our claim follows from Lemma 6.5 in connection with Corollary 2.7. $\diamond$

6.2.2. The case H is of type $\mathcal{C}_2$. In this paragraph we consider the case where H is a $\mathcal{C}_2$-type subgroup of G. We assume that H is maximal. The relevant bounds for $|H|$ and $|B|c_G^{-1}$ are given in Table 1. The parameter k in Table 1 denotes the number of direct summands in the imprimitivity decomposition of V, and $n = mk$.

Lemma 6.10. *Let G be one of the groups of Hypothesis 6.3.*

(a) *Let h and bc_G^{-1} be as in Table 1, and suppose that $m = 1$. Then $hc_G/b < 1$ in any of the following cases:*

- (i) *The group G is linear or unitary and $n \geq 5$, $q \geq 3$ or $q = 2$ and $n \geq 7$.*
- (ii) *The group G is symplectic or orthogonal and $n \geq 6$, $q \geq 5$, or $q = 3, 4$ and $n \geq 9$ or $q = 2$ and $n \geq 18$.*

(b) *If H is a maximal $\mathcal{C}_2$-type subgroup of G such that H stabilizes a decomposition of V into n blocks, then H is not the block stabilizer of an imprimitive irreducible KG-module.*

PROOF. (a) In case $G = \mathrm{SU}_n(q)$ our estimate is
$$\frac{hc_G}{b} \leq \frac{(q+1)^{n+1}n!}{(q^2-q+1)q^{n(n-1)/2}(q^2-1)^{(n-3)/2}} =: f(n,q).$$
For fixed $q \geq 2$, the function $f(n,q)$ is decreasing in n. This is easily verified by considering the quotient $f(n+1,q)/f(n,q)$ of two consecutive terms. Now $f(5,q) < 1$ for all $q \geq 3$ and thus $f(n,q) < 1$ for all $q \geq 3$ and $n \geq 5$. Also, $f(7,2) < 1$ and thus $f(n,2) < 1$ for all $n \geq 7$. The case $G = \mathrm{SL}_n(q)$ is treated similarly. We omit the details.

Suppose now that G is orthogonal. Here, $h = 2^n n!$, and, using the smallest of the three lower bounds for the orders of the Borel subgroups, our estimate is
$$hc_G/b < \frac{2^{n+1}n!}{(q-1)^{(n-1)/2}q^{(n-1)^4/4}} =: f(n,q).$$
For fixed $q \geq 3$ the estimate $f(q,n)$ is a decreasing function for $n \geq 6$ (in fact for $n \geq 4$). Since $f(6,q) < 1$ for all $q \geq 5$, we have $f(n,q) < 1$ for all $n \geq 6$ and $q \geq 5$. Also, $f(9,3), f(9,4) < 1$ and and hence $f(n,q) < 1$ for all $n \geq 9$ and $q = 3,4$. Finally, $f(18,2) < 1$ and our claim follows. The case $G = \mathrm{Sp}_n(q)$ is treated similarly.

(b) We begin with the case $G = \mathrm{SU}_n(q)$. For $n = 3,4$ we use the exact values for $|H|$, $|B|$ and c_G to prove that $|H|c_G/|B| < 1$ for all values of q not excluded in Hypothesis 6.3. By (a), it remains to consider the cases $q = 2$ and $3 \leq n \leq 6$, of which only the case $(n,q) = (5,2)$ is not excluded in Hypothesis 6.3. The latter case is ruled out using the character table of $\mathrm{SU}_5(2)$ available in GAP [**37**]. This concludes the case where G is unitary. The case of $G = \mathrm{SL}_n(q)$ is treated similarly. Note that the case $G = \mathrm{Sp}_n(q)$ does not occur.

Suppose now that G is orthogonal. This case only occurs when q is odd (see [**69**, Table 3.5]), which we assume from now on. By (a)(ii), we only have to consider the cases $q = 3$ and $n = 7, 8$, the first of which is excluded in Hypothesis 6.3. The cases $q = 3$ and $n = 8$ are treated by using the exact values for the orders of B and H (notice that $G = \Omega_8^+(3)$ in this case; see [**69**, Table 3.5.F]). ◇

LEMMA 6.11. *Let G be one of the groups of Hypothesis 6.3. If H is a maximal $\mathcal{C}_2$-type subgroup of G such that H stabilizes a decomposition of V into k blocks with $k > 2$, then $[G:H] > [G:B]c_G$.*

PROOF. We fix $k \geq 3$. First assume that G is one of the groups listed in Lemma 6.10(a). We show $hc_G/b < 1$ by induction on m. By this lemma, this assertion is true for $m = 1$. Now assume that the assertion holds for all $1 \leq m \leq j$ for some $j \geq 1$. Setting $e = 1$ if G is

linear or unitary and $e = 2$, otherwise, we see that increasing m by 1 increases h by at most a factor of

$$\Delta_h := q^{(2km+2k)/e}.$$

On the other hand, the estimate for b increases by at least a factor of

$$\Delta_b := (q-1)^{k/2} q^{k(2km+k-2)/2e}.$$

One easily checks that $\Delta_h/\Delta_b \leq 1$ for all $k \geq 3$ and $m \geq 2$, implying our claim.

To finish the proof, we have to consider the groups satisfying our hypothesis ($k \geq 3$), that are excluded in Lemma 6.10(a). First assume that $G = \mathrm{Sp}_n(q)$, Since $k \geq 3$, we have $n \geq 6$. If $n = 6$, only $k = 3$, $m = 2$ is possible. In this case, $h/b < 1$ for all $q \geq 3$. So we are left with the case $q = 2$, $6 \leq n \leq 18$. Here, h is smallest for $m = 2$. In this case,

$$h/b \leq \frac{(n/2)!}{2^{n(n-6)/4}} < 1$$

for all $n \geq 8$. As $\mathrm{Sp}_6(2)$ is excluded by Hypothesis 6.3, we are left with the case $G = \mathrm{Sp}_8(2)$ and $m = 2$, which is ruled out using the exact value for $|H|$.

Now assume that G is not symplectic. By Lemma 6.10(b), we may assume that $m \geq 2$, and thus $n \geq 6$. As $\mathrm{SU}_6(2)$ is excluded by Hypothesis 6.3, we are left with $G = \mathrm{SL}_6(2)$ or G orthogonal. The former case yields $h/b < 1$. Now suppose that G is orthogonal with $n \geq 8$ even. Here, the bound for h is largest if $m = 2$. Using the smallest valued for b, we obtain

$$\frac{h}{b} \leq \frac{2(n/2)!}{q^{n(n-6)/4}} < 1$$

for all $n \geq 10$. Now $\Omega_8^+(2)$ is excluded by Hypothesis 6.3, and $\Omega_8^-(2)$ is ruled out using its ordinary character table. If G is orthogonal and n odd, we only have to consider the case $G = \Omega_9(3)$, and $m = k = 3$, which again yields $h/b < 1$. The proof is complete. $\diamond$

Lemma 6.12. *Let G be one of the groups of Hypothesis 6.3. If H is a maximal $\mathcal{C}_2$-type subgroup of G such that H stabilizes a decomposition of the natural module of G into two blocks, then there exists an element $t \in G \setminus H$ such that $t \in C_G(H \cap H^t)$. Moreover, if q is odd, there exists such a t of odd order.*

Proof. We first observe that H stabilizes a pair of complementary and isometric subspaces X and Y of V and moreover there exists an involution $s \in \mathrm{GL}(V)$ which normalizes H and interchanges the two

TABLE 1. Bounds for $\mathcal{C}_2$-type groups

G	bc_G^{-1}	$\tilde{H}$	h
$\mathrm{SL}_n(q)$	$q^{n(n-1)/2}(q-1)^{n-1}$	$\mathrm{GL}_m(q) \wr S_k$	$q^{km^2}k!$
$\mathrm{SU}_n(q)$	$\frac{q^2-q+1}{q^2-1}q^{n(n-1)/2}(q^2-1)^{[n/2]}(q+1)^{-1}$	$\mathrm{GU}_m(q) \wr S_k$	$(q+1)^k q^{k(m^2-1)}k!$
$n \geq 3$		$\mathrm{GL}_{n/2}(q^2).2$	$2q^{n^2/2}$
$\mathrm{Sp}_n(q)$	$q^{n^2/4}(q-1)^{n/2}$	$\mathrm{Sp}_m(q) \wr S_k$	$q^{km(m+1)/2}k!$
$n \geq 4$		$\mathrm{GL}_{n/2}(q).2$	$2q^{n^2/4}$
$\Omega_n(q)$	$\frac{1}{2}q^{(n-1)^2/4}(q-1)^{(n-1)/2}$	$\mathrm{GO}_m(q) \wr S_k$	$2^k q^{km(m-1)/2}k!$
$2 \nmid qn$		$\mathrm{GO}_1(p) \wr S_n$	$2^n n!$
$\Omega_n^+(q)$	$\frac{1}{2}q^{n(n-2)/4}(q-1)^{n/2}$	$\mathrm{GO}_m^\epsilon(q) \wr S_k$	$2^k q^{km(m-1)/2}k!$
$n \geq 8$		$\mathrm{GO}_1(q) \wr S_n$	$2^n n!$
		$\mathrm{GL}_{n/2}(q).2$	$2q^{n^2/4}$
		$\mathrm{GO}_{n/2}(q)^2$	$q^{n(n-2)/4}$
$\Omega_n^-(q)$	$\frac{1}{2}q^{n(n-2)/4}(q-1)^{(n-2)/2}(q+1)$	$\mathrm{GO}_m^\epsilon(q) \wr S_k$	$2^k q^{km(m-1)/2}k!$
$n \geq 8$		$\mathrm{GO}_1(q) \wr S_n$	$2^n n!$
		$\mathrm{GO}_{n/2}(q)^2$	$q^{n(n-2)/4}$

subspaces X and Y. To prove our lemma we achieve the hypotheses of Lemma 2.3.

With respect to a suitable basis $\{x_1, \ldots, x_m\}$ for X and $\{y_1, \ldots, y_m\}$ for Y we chose

$$s = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

unless q is odd, G is symplectic, and X and Y are totally singular, in which case we choose

$$s_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If $q = 2, 3$, we put $\mathbb{F}' := \mathbb{F}_{q^2}$. Otherwise, let $\mathbb{F}' := \mathbb{F}_q$. For any subspace $W \leq V$, write $W' := W \otimes_{\mathbb{F}_q} \mathbb{F}'$ and view $\mathrm{GL}(V)$ as a subgroup of $\mathrm{GL}(V')$. For $q \neq 2, 3$ we choose α to be a primitive element in the prime field of $\mathbb{F}_q$. For $q = 2, 3$, we choose $\alpha \in \mathbb{F}'$ of order 3 and 8, respectively. Then $\alpha \neq \alpha^{-1}$.

Let $z \in \mathrm{GL}(V')$ be the element that acts as αI on X' and as $\alpha^{-1}I$ on Y' i.e., with respect to any pair of bases of X' and Y' we have

$$z = \begin{pmatrix} \alpha I & 0 \\ 0 & \alpha^{-1} I \end{pmatrix}.$$

Evidently $H \leq \langle C_G(z), s\rangle$, respectively $H \leq \langle C_G(z), s_1\rangle$ and so we have achieved part of the hypothesis of Lemma 2.3. To find the required element $t \in C_G(s)$, respectively $t \in C_G(s_1)$, we need to further analyze the centralizers of s, respectively s_1 in G, for our possible groups G.

Our strategy is to fix X, Y and s, respectively s_1 and to adjust the form defining G in such a way as to guarantee that our s respectively s_1 and t lie in G and X, Y are non-degenerate or totally singular.

We observe that in the x_i, y_i basis above, the elements of $C_{\mathrm{GL}(V')}(s)$ have the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

and the elements of $C_{\mathrm{GL}(V')}(s_1)$ have the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Next we exhibit elements $t \in C_G(s)$, respectively $t \in C_G(s_1)$ such that $t \in \langle z, z^t\rangle$ for the various situations described above.

If G is linear let

$$t = \begin{pmatrix} I & N_{m,1} \\ N_{m,1} & I \end{pmatrix},$$

where $N_{i,j} := (n_{p,q})$ and $n_{p,q} = \delta_{i,p}\delta_{q,j}$. As $N_{m,1}^2 = 0$ we have

$$t^k = \begin{pmatrix} I & kN_{m,1} \\ kN_{m,1} & I \end{pmatrix},$$

for all integers k and hence t is an element of prime order, equal to the characteristic of G. We also find that $[z^{-1}, z^t] = t^k$ with $k = 2 - \alpha^2 - \alpha^{-2}$. As $2 - \alpha^2 - \alpha^{-2}$ is not equal to 0 and lies in the prime field of $\mathbb{F}'$, we conclude that $t \in \langle z, z^t\rangle$. Clearly, t commutes with s, and thus the first two hypotheses of Lemma 2.3 are satisfied.

To deal with the unitary case, we assume that $n = 2m \geq 4$, and define the $m \times m$ matrix

$$N(a,b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & -b^q & a^q \end{pmatrix},$$

where $a, b \in \mathbb{F}_{q^2}$ such that $a + a^q = 0$ and $a^{q+1} + b^{q+1} = 0$. One checks that $N(a,b)N(a^q,b^q)^T = 0 = N(a,b) + N(a^q,b^q)^T$ where A^T

denotes the transpose of A. In particular this implies that $N(a,b)^2 = -N(a,b)N(a^q,b^q)^T = 0$. We put

$$t = \begin{pmatrix} I & N(a,b) \\ N(a,b) & I \end{pmatrix}$$

and find $t \in \langle z, z^t \rangle$, as in the case $G = \mathrm{SL}_n(q)$.

Finally we pick the defining unitary form of $G = \mathrm{SU}_n(q)$ to be represented by the identity matrix respectively the matrix s depending on whether X and Y are to be non-degenerate respectively totally singular subspaces of V, and observe that in both cases $s, t \in G$.

If $G = \mathrm{Sp}_n(q)$ is symplectic, $n = 2m \geq 4$ and X and Y are non-degenerate, then we define t as in the linear case, that is

$$t = \begin{pmatrix} I & N_{m,1} \\ N_{m,1} & I \end{pmatrix}.$$

Here we pick the defining symplectic form of G to be represented by the matrix

$$\begin{pmatrix} \tilde{J} & 0 \\ 0 & \tilde{J} \end{pmatrix}.$$

To see that $t \in G$ we observe that

$$\begin{aligned} \begin{pmatrix} I & N_{m,1} \\ N_{m,1} & I \end{pmatrix} \begin{pmatrix} \tilde{J} & 0 \\ 0 & \tilde{J} \end{pmatrix} \begin{pmatrix} I & N_{m,1} \\ N_{m,1} & I \end{pmatrix}^T &= \\ \begin{pmatrix} \tilde{J} & N_{m,1}\tilde{J} \\ N_{m,1}\tilde{J} & \tilde{J} \end{pmatrix} \begin{pmatrix} I & N_{m,1}^T \\ N_{m,1}^T & I \end{pmatrix} &= \\ \begin{pmatrix} \tilde{J} + N_{m,1}\tilde{J}N_{m,1}^T & \tilde{J}N_{m,1}^T + N_{m,1}\tilde{J} \\ N_{m,1}\tilde{J} + \tilde{J}N_{m,1}^T & N_{m,1}\tilde{J}N_{m,1}^T + \tilde{J} \end{pmatrix}&. \end{aligned}$$

As $N_{m,1}\tilde{J}N_{m,1}^T = 0 = \tilde{J}N_{m,1}^T + N_{m,1}\tilde{J}$ we have our desired conclusion.

If $G = \mathrm{Sp}_n(q)$ is symplectic, $n = 2m \geq 4$ and X and Y are totally singular, then we define t as

$$t = \begin{pmatrix} I & N_{m,1} \\ -N_{m,1} & I \end{pmatrix}.$$

The commutator $[z^{-1}, z^t]$ shows once more that $t \in \langle z, z^t \rangle$. Again we check that $s_1, t \in G$, where the form defining G is represented by the matrix $\tilde{J}_{2m}$.

If G is orthogonal and $n = 2m \geq 8$, then we define t as

$$t = \begin{pmatrix} I & N \\ N & I \end{pmatrix},$$

where $N = N_{m-1,1} - N_{m,2}$. Now $N^2 = 0$ as $m \geq 4$, and we conclude as before that $t \in \langle z, z^t \rangle$.

To deal with the case that X and Y are totally singular, we define the quadratic form Q on V via the formula $Q(v) = \sum_{i=1}^{m} a_i b_{m-i+1}$ for $v \in V$ with $v = \sum_{i=1}^{m} a_i x_i + b_i y_i$. To deal with the case that X and Y are non-degenerate, we define the quadratic form Q of plus-type on V via the formula $Q(v) = a_1 a_m + a_2 a_{m-1} + \sum_{i=1}^{m} a_i a_{m-i+1} + b_i b_{m-i+1}$ for $v \in V$ with $v = \sum_{i=1}^{m} a_i x_i + b_i y_i$. (As X, Y are both totally singular or both non-degenerate, the form Q is necessarily of plus-type.)

Then $s \in G$. Moreover for our t and $v \in V$ we have $Q(vt) = Q(v)$. Indeed, $vt = (a_1 + b_{m-1})x_1 + (a_2 - b_m)x_2 + a_3 x_3 + \cdots + a_m x_m + (b_1 + a_{m-1})y_1 + (b_2 - a_m)y_2 + b_3 y_3 + \cdots + b_m y_m$. Now if X, Y are non-degenerate, we have $Q(vt) = (a_1 + b_{m-1})a_m + (a_2 - b_m)a_{m-1} + \sum_{i=3}^{m} a_i a_{m-i+1} + (b_1 + a_{m-1})b_m + (b_2 - a_m)b_{m-1} + \sum_{i=3}^{m} b_i b_{m-i+1} = \sum_{i=1}^{m} a_i a_{m-i+1} + b_i b_{m-i+1} = Q(v)$. The calculation for the case X, Y totally singular is very similar. Our claim is now proved.

Put $C := C_G(z)$. We finally show that $C^t \cap Cs = \emptyset$ in each case. The elements of C are of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and thus the elements of Cs are of the form

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

Now

$$t = \begin{pmatrix} I & N \\ N & I \end{pmatrix},$$

for a matrix N with $N^2 = 0$, and thus the elements of C^t are of the form

$$\begin{pmatrix} A - NBN & AN - NB \\ BN - AN & B - NAN \end{pmatrix}.$$

If such an element also lies in Cs, then $A - NBN = 0$ and $B - NAN = 0$. This implies $AN = NA = 0$ and $BN = NB = 0$, a contradiction. A similar calculation shows $C^t \cap Cs_1 = \emptyset$ for the t we have chosen in this case. Thus all the hypotheses of Lemma 2.3 are satisfied, which, together with Corollary 2.7 proves our claim.

In all cases we have chosen t to be a unipotent element of G, and thus t has odd order if q is odd. $\diamond$

We summarize the results of this subsection.

Proposition 6.13. *Let G be one of the following groups.*
(a) *A group as in Hypothesis 6.3.*

(b) *A group* $G = \mathrm{Spin}_n^\epsilon(q)$, $n \geq 7$, *with* q *odd,* $\epsilon \in \{-1, 0, 1\}$ *and* $(n, q) \neq (7, 3)$.

Let H *be a non-parabolic maximal subgroup of* G *of* $\mathcal{C}_2$*-type. Then* H *is not the stabilizer of an imprimitivity decomposition of an irreducible* KG*-module.*

PROOF. If G is not a spin group, the result follows from Lemmas 6.11 and 6.12. If G is a spin group, our claim follows from these lemmas together with Corollary 2.7. $\diamond$

6.2.3. The case H is of type $\mathcal{C}_3$. Here we assume that H is a maximal subgroup of G of type $\mathcal{C}_3$. Table 2 shows the possible pairs (G, H) that need to be considered. In this table, the parameter r denotes the degree of the field extension over which H is realized, and $n = mr$. We begin by reducing to the case $r = 2$.

LEMMA 6.14. *Let* G *be one of the groups of Hypothesis 6.3, and let* H *be a maximal* $\mathcal{C}_3$*-type subgroup of* G.

If $r \geq 3$ *or if* $r = 2$ *and both* G *and* H *are orthogonal, then the maximal dimension of an irreducible* KG*-module is smaller than* $[G\colon H]$.

PROOF. The assertion for $r \geq 3$ is easily established by showing $|H|c_G/|B| \leq 1$, using the bounds given in Table 2.

Suppose now that $r = 2$ and that G and H are orthogonal. Then the bounds in Table 2 give $h/b \leq 4/|T|$ where T is a maximal split torus of G. As $n \geq 7$, we again obtain $|H|c_G/|B| \leq 1$ if $q > 2$. Now suppose that $2 = q$. In this case n is divisible by 4 (see [**69**, Tables 3.5]), and $H \leq \mathrm{GO}_{n/2}^\epsilon(4){:}2$, if $G = \Omega_n^\epsilon(2)$, $\epsilon \in \{1, -1\}$. One easily proves that $|H|^2 < |G|$ if $n \geq 12$, and thus we are done with Lemma 2.1. The result for $n = 8$ (and $G = \Omega_8^-(2)$) is derived form the character table given in the Atlas [**21**, p. 88]. $\diamond$

We now make a few general remarks about $\mathcal{C}_3$-type subgroups where $r = 2$. We begin with an m-dimensional $\mathbb{F}_{q^2}$-vector space V and basis $v_1, \ldots, v_m$. Let $x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then $1, x$ is an $\mathbb{F}_q$ basis of $\mathbb{F}_{q^2}$ and $v_1, xv_1, v_2, xv_2, \ldots, v_m, xv_m$ is an $\mathbb{F}_q$-basis of V. In the following, elements of $\mathrm{Aut}_{\mathbb{F}_q}(V) = \mathrm{GL}(V)$ are written with respect to this basis, unless explicitly stated otherwise. Now assume that $x^{q+1} = 1$ and define $a \in \mathbb{F}_q$ by $x^2 = ax - 1$. Then the polynomial $y^2 - ay + 1$ is irreducible over $\mathbb{F}_q$ (with roots x and x^q).

Let

$$z_0 := \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$$

TABLE 2. Bounds for $\mathcal{C}_3$-type groups; $n = mr$, $r \geq 2$

G	bc_G^{-1}	$\tilde{H}$	h
$\mathrm{SL}_n(q)$	$q^{n(n-1)/2}(q-1)^{n-1}$	$\mathrm{GL}_m(q^r).r$	rq^{rm^2}
$\mathrm{SU}_n(q)$ $r \geq 3$	$\frac{q^2-q+1}{q^2-1}q^{n(n-1)/2}(q^2-1)^{[n/2]}(q+1)^{-1}$	$\mathrm{GU}_m(q^r).r$	$r(q+1)q^{rm^2-1}$
$\mathrm{Sp}_n(q)$ $n \geq 4$	$q^{n^2/4}(q-1)^{n/2}$	$\mathrm{Sp}_m(q^r).r$ $\mathrm{GU}_{n/2}(q).2$	$rq^{rm(m+1)/2}$ $2(q+1)q^{n^2/4-1}$
$\Omega_n(q)$ $2 \nmid qn$	$\frac{1}{2}q^{(n-1)^2/4}(q-1)^{(n-1)/2}$	$\mathrm{GO}_m(q^r).r$	$2rq^{rm(m-1)/2}$
$\Omega_n^+(q)$ $n \geq 8$	$\frac{1}{2}q^{n(n-2)/4}(q-1)^{n/2}$	$\mathrm{GU}_{n/2}(q)$ $\mathrm{GO}_m^+(q^r).r$ $\mathrm{GO}_{n/2}(q^2)$	$(q+1)q^{n^2/4-1}$ $2rq^{rm(m-1)/2}$ $2q^{n(n-2)/4}$
$\Omega_n^-(q)$ $n \geq 8$	$\frac{1}{2}q^{n(n-2)/4}(q-1)^{(n-2)/2}(q+1)$	$\mathrm{GU}_{n/2}(q)$ $\mathrm{GO}_m^-(q^r).r$ $\mathrm{GO}_{n/2}(q^2)$	$(q+1)q^{n^2/4-1}$ $2rq^{rm(m-1)/2}$ $2q^{n(n-2)/4}$

and observe that the minimal polynomial of z_0 equals $y^2 - ay + 1$, and that z_0 is the matrix representing right muliplication with x in $\mathbb{F}_{q^2}$ with respect to the basis $1, x$.

Define z to be the $(n \times n)$-block diagonal matrix all of whose blocks are equal to z_0. We have the following well known fact.

LEMMA 6.15. *If $t \in \mathrm{GL}(V)$ and M is the matrix representing t with respect to the basis $v_1, xv_1, v_2, xv_2, \ldots, v_m, xv_m$, then t is $\mathbb{F}_{q^2}$-linear if and only if $zM = Mz$.*

LEMMA 6.16. *Let G be one of the groups of Hypothesis 6.3. Suppose that H is a maximal $\mathcal{C}_3$-type subgroup of G defined over a field extension of $\mathbb{F}_q$ of degree 2, and that H is not orthogonal if G is orthogonal. Then there exists an element $t \in G \setminus H$ such that $t \in C_G(H \cap H^t)$. Moreover, if G is orthogonal and q is odd, there exists such a t of odd order.*

PROOF. Notice that Lemma 6.14 excludes the case that G is unitary. By Lemma 6.15 we may assume $H = N_G(\langle z \rangle)$. Moreover, if $s \in \mathrm{GL}_n(q)$ with $z^s = z^{-1}$ and $G^s = s$, then s induces a field automorphism of order two on $C_G(z)$ and $H = N_G(\langle z \rangle) = \tilde{H} \cap G$ with $\tilde{H} := \langle \tilde{C}, s \rangle$ where $\tilde{C} := C_{\mathrm{GL}_n(q)}(z)$.

Put

$$s_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let s be the $(n \times n)$-block diagonal matrix all of whose blocks are equal to s_0. Then $z^s = z^{-1}$. As in the previous subsection we see that any matrix that commutes with s is a block matrix all of whose blocks must have the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with $a, b \in \mathbb{F}_q$.

Suppose first that G is linear, i.e., $G = \mathrm{SL}_n(q)$. Then $G^s = G$ and thus $H = \tilde{H} \cap G$. Define t to be a block diagonal matrix with diagonal blocks t_0 and I_{n-4} where

$$t_0 = \begin{pmatrix} I_2 & 0 \\ s_0 & I_2 \end{pmatrix}.$$

Then $t \in G \setminus H$ and t commutes with s. A straightforward calculation shows that the upper left (4×4)-block of $[z^{-1}, z^t]$ equals

$$\begin{pmatrix} I_2 & 0 \\ s_0(2I_2 - z_0^{-2} - z_0^2) & I_2 \end{pmatrix}.$$

Now $2I_2 - z_0^{-2} - z_0^2 = (4 - a^2)I_2$ and thus $[z^{-1}, z^t] = t^{4-a^2}$. As $a \neq \pm 2$ and t is of prime order, it follows that $t \in \langle z, z^t \rangle$. So the first two hypotheses of Lemma 2.3 are satisfied. We now verify its third hypothesis. We view the elements of $\tilde{C}$ as block matrices $(A_{ij})_{1 \leq i,j \leq m}$, where each block A_{ij} is a (2×2)-matrix commuting with z_0. Let (A_{ij}) be an element of $\tilde{C}$. Then the first row of (2×2)-blocks of $(A_{ij})^t$ consists of the matrices $A_{1,1} - A_{1,2}s_0, A_{1,2}, \ldots, A_{1,m}$. On the other hand, the first row of $(A_{ij})s$ consists of the matrices $A_{1,1}s_0, A_{1,2}s_0, \ldots, A_{1,m}s_0$. One easily checks that a (2×2)-matrix M commuting with z_0 and satisfying $M = Ms_0$ is the zero matrix or $a = -2$. Since $a \neq \pm 2$, the latter case cannot occur. Thus $\tilde{C}^t \cap \tilde{C}s = \emptyset$. It follows from Lemma 2.3 that t commutes with $\tilde{H} \cap \tilde{H}^t$, hence t commutes with $H \cap H^t$.

Now let G be symplectic, i.e., $G = \mathrm{Sp}_n(q)$. In this case H is either symplectic or unitary. To deal with the case that H is unitary, let F be the block diagonal matrix with blocks

$$\tilde{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and assume that F represents the form defining G. Then $z \in G$ and thus $C_G(z)$ is unitary, and we may take $H = N_G(\langle z \rangle)$.

In the following, we choose a non-degenerate, $2k$-dimensional, z-invariant subspace X of V, e.g., $X := \langle v_1, xv_1, \ldots, v_k, xv_k \rangle$ in the notation introduced before Lemma 6.15, and put $S := \{\alpha \in \mathrm{Stab}_G(X) \mid \alpha_{X^\perp} = I_{X^\perp}\}$. Then $S \cong \mathrm{Sp}(X) \cong \mathrm{Sp}_{2k}(q)$. We view z_X as an element of S.

First assume that $q > 3$. Here, we let $k = 1$, and thus $S \cong \mathrm{SL}_2(q)$. Put $N := N_S(\langle z_0 \rangle)$. By Lemma 2.9, there is an element $t \in S$ such that t centralizes $N \cap N^t = Z(S)$ and $\langle z_0, z_0^t \rangle = S$. As S is perfect, we find that S is the derived subgroup of $\langle z, z^t \rangle$. Now $H \cap H^t \leq N_G(\langle z, z^t \rangle)$, and thus $H \cap H^t$ normalizes S. It follows that $H \cap H^t \leq N_G(S) = \mathrm{Sp}(X) \times \mathrm{Sp}(X^\perp)$. By construction, $t \in C_G(H \cap H^t)$.

When $q = 3$, we let $k = 2$. Then $S \cong \mathrm{Sp}(X) \cong \mathrm{Sp}_4(3)$. We note that z_X corresponds to an element of order 4 in $\mathrm{Sp}_4(3)$ which squares to $-I_4$. Using GAP, we find an element t of order 5 in S which is inverted by z_X, and such that $t \in C_S(N_S(\langle z_X \rangle) \cap N_S(\langle z_X \rangle)^t)$. The first of these properties implies that $t \in \langle z, z^t \rangle$, and that t generates a normal Sylow 5-subgroup of $\langle z, z^t \rangle$. As an element of order 5 of S clearly does not fix any non-trivial vector of X, the fixed space of $\langle t \rangle$ equals $X^\perp$. Thus $H \cap H^t \leq N_G(\langle z, z^t \rangle) \leq N_G(\langle t \rangle)$ fixes $X^\perp$. The second property of t now implies that $t \in C_G(H \cap H^t)$.

Finally suppose that $q = 2$. This time we let $k = 3$, which is possible as $n \geq 8$ by Hypothesis 6.3. Then $S \cong \mathrm{Sp}(X) \cong \mathrm{Sp}_6(2)$. Using GAP, we find an element $t \in S$ such that $\langle z_X, z_X^t \rangle$ is isomorphic to the simple group $\mathrm{SL}_8(2)$. Moreover, t can be chosen such that $t \in C_S(N_S(\langle z_X \rangle) \cap N_S(\langle z_X \rangle)^t)$. (In fact t can be chosen to be an involution whose centralizer in S has order 384.) These choices imply that the derived subgroup of $\langle z, z^t \rangle$ is equal to $\langle z_X, z_X^t \rangle \cong \mathrm{SL}_8(2)$. The maximal subgroup $\mathrm{SL}_2(8)\colon 3$ of $\mathrm{Sp}_6(2)$ acts irreducibly on the natural 6-dimensional module of $\mathrm{Sp}_6(2)$, and the restriction of this module to $\mathrm{SL}_2(8)$ is also irreducible. It follows that the fixed space of $\langle z_X, z_X^t \rangle$ on V equals $X^\perp$. Hence $H \cap H^t \leq N_G(\langle z, z^t \rangle) \leq N_G(\langle z_X, z_X^t \rangle)$ stabilizes $X^\perp$. In turn, this implies that $t \in C_G(H \cap H^t)$.

We now consider the case where H is symplectic. In this case $n = 2m = 4k$ and H is of type $\mathrm{Sp}_{2k}(q^2).2$. We choose the $\mathbb{F}_{q^2}$-bilinear symplectic form on V stabilized by H in such a way that its Gram matrix with respect to our $\mathbb{F}_{q^2}$-basis $\{v_1, \ldots, v_m\}$ is the block diagonal matrix whose blocks are

$$\tilde{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Composing this $\mathbb{F}_{q^2}$-bilinear form with the trace $\mathbb{F}_{q^2} \to \mathbb{F}_q$, we obtain an $\mathbb{F}_q$-bilinear symplectic form on the $2m$-dimensional $\mathbb{F}_q$-vector space V. We take G to be the group stabilizing this latter from. With respect to

the $\mathbb{F}_q$-basis $\{v_1, xv_1, \ldots, v_m, xv_m\}$, the Gram matrix of this latter form is a block diagonal matrix consisting of k blocks of the (4×4)-matrix E_0 with

$$E_0 := \begin{pmatrix} 0 & 0 & 2 & a \\ 0 & 0 & a & a^2-2 \\ -2 & -a & 0 & 0 \\ -a & -a^2+2 & 0 & 0 \end{pmatrix}.$$

To simplify our calculations we change our $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$, resulting in a change of the $\mathbb{F}_q$ basis of V.

Recall that the minimal polynomial of z_0 is $y^2 - ay + 1$. Whenever possible, we choose z_0 such that $a \notin \{0, 1\}$. Such a choice is possible if $q \neq 2, 3, 5$. When $q = 2, 5$ we choose $a = 1$ and when $q = 3$ we choose $a = 0$. If $q \neq 2, 3, 5$ the base change matrix we use is

$$B_0 := \begin{pmatrix} 1 & u \\ -u & u \end{pmatrix}$$

where $u = -(1-a)^{-1} = -(a-1)^{-1}$. Note that $B_0 z_0 = z_0 B_0$ and that the new Gram matrix has block diagonal form with each block equal to

$$F_0 := (1+u)\begin{pmatrix} 0 & 0 & 1-u & -2u \\ 0 & 0 & -2u & 1-u \\ u-1 & 2u & 0 & 0 \\ 2u & u-1 & 0 & 0 \end{pmatrix}.$$

If $q = 2, 5$ the base change matrix we use is

$$B_0 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = z_0,$$

and the new Gram matrix has block diagonal form where each block is equal to

$$F_0 := \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}.$$

If $q = 3$ the base change matrix we use is

$$B_0 := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Again we note that $B_0 z_0 = z_0 B_0$ and that the new Gram matrix has block diagonal form where each block is equal to

$$F_0 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We now view s as an element of $\mathrm{GL}(V)$, written with respect to our new $\mathbb{F}_q$-basis of V. Then $s \in G$ and $H = N_G(\langle z \rangle) = \langle C_G(z), s \rangle$. Let $t \in \mathrm{GL}(V)$ be the element represented, with respect to the new basis, by the block diagonal matrix $\mathrm{diag}(t_0, I_{n-4})$ with

$$t_0 = \begin{pmatrix} I_2 & 0 \\ s_0 & I_2 \end{pmatrix}.$$

Then $t \in G \setminus H$. As in the linear case we find that the hypotheses of Lemma 2.3 are satisfied.

We now consider the cases where G is orthogonal. Here H is either orthogonal or unitary. The first of these cases is excluded by the hypothesis of the lemma, so let us assume that H is unitary.

We may assume that the quadratic space V is the orthogonal direct sum of the 2-dimensional, z-invariant, anisotropic spaces $V_1, \ldots, V_m$, with $V_i := \langle v_i, xv_i \rangle$, where the form on V_i is given by $(\alpha, \beta) \mapsto \alpha^2 + a\alpha\beta + \beta^2$ with respect to the basis $\{v_i, xv_i\}$. Let $V = X \oplus Y$ with $X := V_1 \oplus \cdots \oplus V_4$ and $Y := V_5 \oplus \cdots \oplus V_m$. Then X is a non-degenerate 8-dimensional subspace X of plus-type, invariant under z, and $Y = X^\perp$. Our aim is to produce an element $t \in G$ acting trivially on Y, such that $t \in C_G(H \cap H^t)$.

We choose a new basis $w_1, \ldots, w_8$ of X such that (w_i, w_{9-i}) is a hyperbolic pair for $1 \leq i \leq 4$, such the totally isotropic subspaces $X_0 := \langle w_1, \ldots, w_4 \rangle$ and $X_0' := \langle w_5, \ldots, w_8 \rangle$ of X are z-invariant, and such that the matrix of z_X with respect to $\{w_1, \ldots, w_8\}$ equals $\mathrm{diag}(z_0, z_0, z_0^*, z_0^*)$ with

$$z_0^* = J_2^{-1} z_0^{-T} J_2 = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}.$$

To see that such a basis of X exists, let $w_1' := (1, 0, \alpha, \beta)$ and $w_2' := (0, 1, -\beta, \alpha + a\beta)$ be elements of $V_1 \oplus V_2$, written with respect to the basis $\{v_1, xv_1, v_2, xv_2\}$ of $V_1 \oplus V_2$. Here, $\alpha, \beta \in \mathbb{F}_q$ satisfy $\alpha^2 + a\alpha\beta + \beta^2 = -1$. Then $\langle w_1', w_2' \rangle$ is a totally isotropic subspace of $V_1 \oplus V_2$, and $w_1' z = w_2'$ and $w_2' z = -w_1' + a w_2'$. Next, choose an isotropic vector $w_3' \in V_1 \oplus V_2$, orthogonal to w_1' and with scalar product 1 with w_2'. In other words, (w_2', w_3') is a hyperbolic pair of $V_1 \oplus V_2$, orthogonal to w_1'. By subtracting a suitable scalar multiple of w_1', we may assume that w_3' is of the

form $w_3' = (0, \gamma, \delta, \varepsilon)$. Now put $w_4' := -w_3' z$. Then (w_1', w_4') is a hyperbolic pair orthogonal to (w_2', w_3'), and $w_4' z = w_3' + a w_4'$ (notice that $z^2 = az - I_V$). Producing an analogous basis for $V_3 \oplus V_4$ and rearranging the basis vectors, we obtain the desired basis $\{w_1, \ldots, w_8\}$ of X.

By replacing the first eight elements of $\{v_1, xv_1, \ldots, v_m, xv_m\}$ by $\{w_1, \ldots, w_8\}$, we obtain a new basis of V. Elements of $\mathrm{GL}(V)$ are now written with respect to this new basis. Again, we view s as an element of $\mathrm{GL}(V)$, written with respect to the new basis. Then $s \in G$ and $H = N_G(\langle z \rangle) = \langle C_G(z), s \rangle$.

Now assume that $q \neq 3$. Then $a \neq 0$. Define $t_0 \in \Omega(X)$ by the matrix

$$\begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ R & 0 & I_2 & 0 \\ 0 & -R & 0 & I_2 \end{pmatrix},$$

with

$$R := \begin{pmatrix} -2a^{-1} & 1 \\ 1 & -2a^{-1} \end{pmatrix}.$$

The cyclic group $Z := \langle \mathrm{diag}(z_0, z_0, z_0^*, z_0^*) \rangle$ of order $q+1$ acts linearly on the 4-dimensional space of matrices

$$U := \left\{ \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ S & 0 & I_2 & 0 \\ 0 & S' & 0 & I_2 \end{pmatrix} \mid S \in \mathbb{F}_q^{2\times 2}, S' = -J_2 S^T J_2 \right\}$$

by conjugation. As an $\mathbb{F}_q Z$-module, U is the direct sum $C_U(Z) \oplus [U, Z]$. The space $C_U(Z)$ is 2-dimensional, hence is $[U, Z]$. It is easy to see that $t_0 \in [U, Z]$, in fact $t_0 = [u, z_X]$ with

$$u := \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ S & 0 & I_2 & 0 \\ 0 & S' & 0 & I_2 \end{pmatrix},$$

where

$$S := \begin{pmatrix} 2a^{-1} & 0 \\ -1 & 0 \end{pmatrix}.$$

Moreover, t_0 and $[t_0, z_X]$ are linearly independent in $[U, Z]$. Thus $[U, Z]$ is an irreducible $\mathbb{F}_q Z$-module. It follows that $\langle z_X, z_X^{t_0} \rangle = \langle z_X, [t_0, z_X] \rangle = \langle z_X, t_0 \rangle = Z[U, Z]$. In particular, $t_0 \in \langle z_X, z_X^{t_0} \rangle$. Also, the derived subgroup of $\langle z_X, z_X^{t_0} \rangle$ equals $[U, Z]$, whose fixed space on X equals X_0. Now let $t \in G$ be the element which acts as t_0 on X and as I_Y on Y.

Then the fixed space of the derived subgroup of $\langle z, z^t \rangle$ equals $X_0 \oplus Y$. As $H \cap H^t \leq N_G(\langle z, z^t \rangle)$, it follows that $H \cap H^t$ fixes $X_0 \oplus Y$ and $X_0 = (X_0 \oplus Y)^\perp$. Now let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an element of $H \cap H^t$, written with respect to our new basis of V, where A is an (8×8)-matrix. Then

$$g^t = \begin{pmatrix} t_0^{-1} A t_0 & t_0^{-1} B \\ C t_0 & D \end{pmatrix}.$$

Since g stabilizes X_0, the first four rows of B are zero, and thus $t_0^{-1} B = B$. Similarly, as g stabilizes $X_0 + Y$, the last four columns of C are zero, and hence $Ct_0 = C$. It also follows that A is invertible, and thus $A \in N_{\mathrm{GL}_8(q)}(\langle z_X \rangle) \cap N_{\mathrm{GL}_8(q)}(\langle z_X^{t_0} \rangle)$. Now $N_{\mathrm{GL}_8(q)}(\langle z_X \rangle) = \langle C_{\mathrm{GL}_8(q)}(z_X), s_X \rangle$, and s_X commutes with t_0. Thus, by Lemma 2.3, in order to prove that A commutes with t_0, we are left to show that $C_{\mathrm{GL}_8(q)}(z_X)^{t_0} \cap C_{\mathrm{GL}_8(q)}(z_X) s_X = \emptyset$. This is done in a similar way as in the proof for the linear case.

Now let $q = 3$. Using GAP, we find that the conjugacy class of z_X in $\mathrm{GO}(X) \cong \mathrm{GO}_8^+(3)$ is characterized by the property that $|z_X| = 4$ and $z_X^2 = -I_X$. Using GAP, one can show that there is an element $t_0 \in \mathrm{GO}(X)$ of order 5, centralizing $N_{\mathrm{GO}(X)}(\langle z_X \rangle) \cap N_{\mathrm{GO}(X)}(\langle z_X^{t_0} \rangle)$, such that $\langle z_X, z_X^t \rangle$ has order 20 and such that its derived subgroup is spanned by t_0 (in particular, $t_0 \in \langle z_X, z_X^{t_0} \rangle$). Moreover, t_0 does not have any non-trivial fixed vector on X. Now let $t \in G$ be defined by letting it act as t_0 on X and as I_Y on Y. Then the fixed space of the derived subgroup of $\langle z, z^t \rangle$ equals Y. As $H \cap H^t \leq N_G(\langle z, z^t \rangle)$, it follows that $H \cap H^t$ stabilizes Y and hence also X. As $(H \cap H^t)_X$ equals $N_{\mathrm{GO}(X)}(\langle z_X \rangle) \cap N_{\mathrm{GO}(X)}(\langle z_X^{t_0} \rangle)$, it follows that t centralizes $H \cap H^t$.

In any case, the element t constructed for G orthogonal and q odd has odd order. $\diamond$

Remark 6.17. As $\mathrm{Sp}_4(2) \cong S_6$ is not simple, this group has been excluded in Lemma 6.16. Nevertheless, there is an overfield subgroup $H = \mathrm{Sp}_2(4).2 \cong S_5$, and our argument in the proof of this lemma shows that S_6 has an element t such that $t \in C_{S_6}(H \cap H^t)$. However A_6 contains no such element.

We summarize the results of this subsection.

Proposition 6.18. *Let G be one of the following groups.*
(a) *A group as in Hypothesis 6.3.*

(b) *A group* $G = \mathrm{Spin}_n^\epsilon(q)$, $n \geq 7$, *with* q *odd,* $\epsilon \in \{-1, 0, 1\}$ *and* $(n, q) \neq (7, 3)$.

Let H *be a non-parabolic maximal subgroup of* G *of* $\mathcal{C}_3$*-type. Then* H *is not the stabilizer of an imprimitivity decomposition of an irreducible* KG*-module.*

Proof. If G is not a spin group, the result follows from Lemmas 6.14 and 6.16. If G is a spin group, our claim follows from these lemmas together with Corollary 2.7. $\diamond$

6.2.4. The case H is of type $\mathcal{C}_4$ or $\mathcal{C}_7$. Here we assume that H is a maximal subgroup of G of type $\mathcal{C}_4$ or $\mathcal{C}_7$. Table 3 gives the relevant estimates for bc_G^{-1} and h.

Lemma 6.19. *Let* G *be one of the groups of Hypothesis 6.3. If* H *is a maximal* $\mathcal{C}_4$*-type subgroup of* G*, then* $hc_G/b \leq 1$.

Proof. We may assume that m and n/m are distinct, since otherwise H is contained in a $\mathcal{C}_7$-type subgroup. Hence $n \geq 6$. Fix n and hence b. We first identify the values of m for which h and consequently hc_G/b is maximal.

If $G \neq \mathrm{Sp}_n(q)$, then we observe that the exponent of q in h is of the form $t(x)+t(n/x)$ where t is quadratic polynomial in x which is positive and increasing on the interval $[2, n-2]$. So by symmetry we see that on the interval $[2, n-2]$ the function $t(x) + t(n/x)$ obtains a minimum at the point $x = n/x$ and maxima at the end points. Evidently the endpoint maxima are equal. If $G = \mathrm{Sp}_n(q)$, the exponent of q in h is of the form $t(x) + s(n/x)$, where $t(x)$ is as before and $s(x) = t(x) - x$. Here we see that the absolute maximum occurs at the right end point, i.e., when $x = n/2$.

For example, if $G = \mathrm{SU}_n(q)$ and $m = n/2$, the exponent of q in h is $n^2/4+2$, whereas the exponent of q in b is $n(n-1)/2$. Thus, as $n \geq 6$, we find

$$\frac{hc_G}{b} \leq \frac{(q^2-1)(q+1)^3 q^{n^2/4+2}}{(q^2-q+1)(q^2-1)^{[n/2]} q^{(n(n-1)/2)}} \leq \frac{1}{q^{(n^2-2n-8)/4}} < 1.$$

Similar calculations for the other cases prove our claim. $\diamond$

The case of $\mathcal{C}_7$-type subgroups is easily reduced to the previous case.

Lemma 6.20. *Let* G *be one of the groups of Hypothesis 6.3. If* H *is a maximal* $\mathcal{C}_7$*-type subgroup of* G*. Then* $|H|c_G/|B| \leq 1$ *unless* $G = \mathrm{Sp}_4(q)$. *In the latter case* H *is a* $\mathcal{C}_1$*-type subgroup (which has been ruled out as a block stabilizer of an imprimitive irreducible* KG*-module in Lemma 6.5).*

TABLE 3. Bounds for $\mathcal{C}_4$-type groups; $n = ms$, $m \neq s$

G	bc_G^{-1}	$\tilde{H}$	h
$\mathrm{SL}_n(q)$	$q^{n(n-1)/2}(q-1)^{n-1}$	$\mathrm{GL}_m(q) \otimes GL_s(q)$	$q^{m^2+s^2}$
$\mathrm{SU}_n(q)$ $n \geq 3$	$\frac{q^2-q+1}{q^2-1} q^{n(n-1)/2}(q^2-1)^{[n/2]}(q+1)^{-1}$	$\mathrm{GU}_m(q) \otimes \mathrm{GU}_s(q)$	$(q+1)^2 q^{m^2-1+s^2-1}$
$\mathrm{Sp}_n(q)$ $n \geq 4$	$q^{n^2/4}(q-1)^{n/2}$	$\mathrm{Sp}_m(q) \otimes \mathrm{GO}_s^{\epsilon}(q)$	$2q^{(m(m+1)+s(s-1))/2}$
$\Omega_n(q)$ $2 \nmid (qn)$	$\frac{1}{2} q^{(n-1)^2/4}(q-1)^{(n-1)/2}$	$\mathrm{GO}_m(q) \otimes \mathrm{GO}_s(q)$	$4q^{(m(m-1)+s(s-1))/2}$
$\Omega_n^+(q)$ $n \geq 8$	$\frac{1}{2} q^{n(n-2)/4}(q-1)^{n/2}$	$\mathrm{Sp}_m(q) \otimes \mathrm{Sp}_s(q)$ $\mathrm{GO}_m^{\epsilon_1}(q) \otimes \mathrm{GO}_s^{\epsilon_2}(q)$	$q^{(m(m+1)+s(s+1))/2}$ $4q^{(m(m-1)+s(s-1))/2}$
$\Omega_n^-(q)$ $n \geq 8$	$\frac{1}{2} q^{n(n-2)/4}(q-1)^{(n-2)/2}(q+1)$	$\mathrm{GO}_m(q) \otimes \mathrm{GO}_s^-(q)$	$4q^{(m(m-1)+s(s-1))/2}$

PROOF. Here, H is the stabilizer of a tensor product decomposition

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_t$$

with m-dimensional subspaces $V_1, \ldots, V_t$, where $t, m \geq 2$. Now consider the maximal $\mathcal{C}_4$-type subgroup H_4 of G stabilizing the tensor decomposition $V = V_1 \otimes W$ with $W = V_2 \otimes \cdots \otimes V_t$.

Let h_4 and h denote the upper bounds for $|H_4|$ and $|H|$ given in Tables 3 and 4, respectively. If $t \geq 3$, then $|H| < h_4$. This is proved by showing $h < h_4$, except in the case $G = \mathrm{Sp}_8(q)$ with $t = 3, m = 2$. In the latter case, we use the exact value of $|\tilde{H}|$ as an upper bound for $|H|$ and the fact that q is odd (see [**69**, Table 3.5.C]). Now $|H|c_G/|B| \leq |H|c_G/b < h_4 c_G/b$, and thus Lemma 6.19 establishes our claim in this case. If $t = 2$ and $m \geq 3$, then $hc_G/b < 1$ and we are done.

If $t = 2 = m$, then $n = 4$ and so G is not orthogonal. If G is symplectic, then G is isomorphic to an orthogonal group and H is the stabilizer of a nonsingular 1-space. If G is unitary or symplectic, we use the exact values of $|\tilde{H}|$ (see Table 4) as upper bounds for $|H|$. These give $|H|c_G/|B| < 1$. $\diamond$

We summarize the results of this subsection.

PROPOSITION 6.21. *Let G be one of the following groups.*

(a) *A group as in Hypothesis 6.3.*

TABLE 4. Bounds for $\mathcal{C}_7$-type groups; $n = m^t$

G	bc_G^{-1}	$\tilde{H}$	h
$\mathrm{SL}_n(q)$	$q^{n(n-1)/2}(q-1)^{n-1}$	$\mathrm{GL}_m(q) \wr S_t$	$q^{tm^2}\ t!$
$\mathrm{SU}_n(q)$ $n \geq 3$	$\frac{q^2-q+1}{q^2-1} q^{n(n-1)/2}(q^2-1)^{[n/2]}(q+1)^{-1}$	$\mathrm{GU}_m(q) \wr S_t$	$(q+1)^t q^{t(m^2-1)}\ t!$
$\mathrm{Sp}_n(q)$ $n \geq 4$	$q^{n^2/4}(q-1)^{n/2}$	$\mathrm{Sp}_m(q) \wr S_t$	$q^{tm(m+1)/2}\ t!$
$\Omega_n(q)$ $2 \mid (qn)$	$\frac{1}{2} q^{(n-1)^2/4}(q-1)^{(n-1)/2}$	$\mathrm{GO}_m(q) \wr S_t$	$2^t q^{tm(m-1)/2}\ t!$
$\Omega_n^+(q)$ $n \geq 8$	$\frac{1}{2} q^{n(n-2)/4}(q-1)^{n/2}$	$\mathrm{Sp}_m(q) \wr S_t$ $\mathrm{GO}_m^\epsilon(q) \wr S_t$	$q^{tm(m+1)/2}\ t!$ $2^t q^{tm(m-1)/2}\ t!$

(b) *A group $G = \mathrm{Spin}_n^\epsilon(q)$, $n \geq 7$, with q odd, $\epsilon \in \{-1, 0, 1\}$ and* $(n, q) \neq (7, 3)$.

Let H be a non-parabolic maximal subgroup of G of $\mathcal{C}_4$-type or $\mathcal{C}_7$-type. Then H is not the stabilizer of an imprimitivity decomposition of an irreducible KG-module.

PROOF. The result follows from Lemmas 6.19 and 6.20. $\diamond$

6.2.5. The case H is of type $\mathcal{C}_5$. Suppose now that H is a maximal subgroup of G of type $\mathcal{C}_5$. Table 5 gives the relevant estimates for b and h.

LEMMA 6.22. *Let G be one of the groups of Hypothesis 6.3. If H is a maximal $\mathcal{C}_5$-type subgroup of G, then $|H|c_G/|B| \leq 1$ unless $G = \mathrm{SU}_n(q)$ and H is symplectic.*

PROOF. Suppose that $G = \mathrm{SL}_n(q)$. We first consider the case $n = 2$. Then $|H| \leq 2|\mathrm{SL}_2(q_0)|$. Hence

$$\frac{|H|}{|B|} \leq \frac{2q_0(q_0^2-1)}{q_0^k(q_0^k-1)} \leq \frac{2}{q_0^{k-1}} < 1$$

if $q_0 \neq 2$ or $q_0 = 2$ and $k \geq 3$. As $\mathrm{SL}_2(4)$ is excluded by Hypothesis 6.3, our claim follows in this case. Now let $n = 3$. Then $|H| \leq 3|\mathrm{SL}_3(q_0)|$. As $3|\mathrm{SL}_3(q_0)|$ is largest for $q_0 = \sqrt{q}$, we find

$$\frac{|H|}{|B|} \leq \frac{3q_0^3(q_0^2-1)(q_0^3-1)}{q_0^6(q_0^2-1)^2} = \frac{3(q_0^2+q_0+1)}{q_0^3(q_0+1)} < 1$$

for all $q_0 \geq 2$. Next, let $n = 4$. Here, we use the obvious bound $|H| \leq |GL_4(q_0)|(q-1)$. Again, we may assume that $q = q_0^2$, and so

$$\frac{|H|}{|B|} \leq \frac{(q_0-1)(q_0^3-1)(q_0^2+1)}{q_0^6} < 1$$

for all $q_0 \geq 2$. Suppose now that $n \geq 5$. Then h is maximal if $q = q_0^2$. In this case

$$\frac{h}{b} = \frac{q_0^n}{(q_0^2-1)^{n-2}} \leq 1$$

if $q_0 \geq 3$ or $q_0 = 2$ and $n \geq 6$. The case $q_0 = 2$ and $n = 5$ can be ruled out using the bound $|H| \leq |\mathrm{GL}_5(2)|(q-1)$.

Let $G = \mathrm{SU}_n(q)$ and H unitary. Then $q = q_0^k$ with $k \geq 3$ (see [**69**, Table 3.5.B]). Clearly, h is maximal if $k = 3$. Suppose first that $n = 3$. Here we use the sharper bound $|H| \leq 3|\mathrm{SU}_3(q_0)|$ and the exact value $|B| = q^3(q^2-1)$ to obtain

$$\begin{aligned}
\frac{|H|c_G}{|B|} &\leq \frac{3q_0^3(q_0^2-1)(q_0^3+1)(q^2-1)}{q^3(q^2-1)(q^2-q+1)} \\
&\leq \frac{3(q_0^2-1)(q_0^3+1)}{q_0^6(q_0^6-q_0^3+1)} \\
&\leq \frac{3 \cdot 3 \cdot 9}{64 \cdot 57} < 1
\end{aligned}$$

for all $q_0 \geq 2$. For $n \geq 4$ we have

$$\begin{aligned}
\frac{hc_G}{b} &= \frac{(q+1)^3}{q_0^{(n^2-3n+2)/2}(q^2-1)^{[n/2]-1}(q^2-q+1)} \\
&\leq \frac{(q_0^3+1)^2}{q_0^{(n^2-3n+2)/2}(q_0^6-1)^{[n/2]-1}} \\
&\leq \frac{(q_0^3+1)^2}{q_0^3(q_0^6-1)} \\
&\leq \frac{9^2}{8 \cdot 7 \cdot 9} < 1
\end{aligned}$$

for all $q_0 \geq 2$.

If $G = \mathrm{SU}_n(q)$ and H is neither unitary nor symplectic then

$$\frac{hc_G}{b} \leq \frac{2(q+1)}{(q^2-1)^{[n/2]-1}} < 1$$

for all $n \geq 4$. For $n = 3$ our claim can be checked by using the exact value for $|B|$.

If $G = \mathrm{Sp}_n(q)$, then h is maximal if $q = q_0^2$ and H is symplectic. In this case

$$\frac{h}{b} = \frac{2q_0^{n/2}}{(q_0^2-1)^{n/2}} \leq \frac{2^{(n+2)/2}}{3^{n/2}} < 1$$

for all $n \geq 4$.

If G is orthogonal, then h is maximal if $q = q_0^2$. In this case

$$\frac{h}{b} = \frac{4q_0^{[n/2]}}{(q_0^2-1)^{[n/2]}} < 1$$

for $n \geq 8$ and $n = 7$, $q_0 \geq 3$, which implies our claim. Our proof is now complete. $\diamond$

Lemma 6.23. *If $G = \mathrm{SU}_n(q)$ and H is a maximal subgroup of symplectic $\mathcal{C}_5$-type, then there exists an element $t \in G \setminus H$ with $t \in C_G(H \cap H^t)$.*

Proof. Here $n = 2m$ and we may define G as

$$G := \{A \in \mathrm{SL}_n(\mathbb{F}_{q^2}) \mid A^T \hat{J} \rho(A) = \hat{J}\},$$

where ρ is the automorphism of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ raising every matrix entry to its qth power, and

$$\hat{J} := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Notice that $\hat{J}$ defines a skew Hermitian form on $\mathbb{F}_{q^2}^n$ if q is odd. In this case we choose $\alpha \in \mathbb{F}_{q^2}^n$ with $\alpha^q = -\alpha$. Then $\alpha\hat{J}$ defines a non-degenerate Hermitian form with isometry group G.

In this setting, we put

$$C := \{A \in G \mid \rho(A) = A\},$$

i.e. $C = C_G(\rho)$. Then $H = N_G(C)$. In fact, $H = \langle C, s\rangle \cap G$ with $s = \beta s_0$, where

$$s_0 := \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$

and $\beta \in \mathbb{F}_{q^2}^*$ of order $2(q+1)$ if q is odd, and of order $q+1$ if q is even. In the latter case, $H = C \times Z(G)$.

We choose

$$t = \begin{pmatrix} A & 0 \\ 0 & \rho(A)^{-T} \end{pmatrix},$$

with

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & I_{m-1} \end{pmatrix},$$

if q is even, where $\lambda \in \mathbb{F}_{q^2}$ is a non-trivial element with $\lambda^{q+1} = 1$, and

$$A = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-2} \end{pmatrix},$$

if q is odd, where $\lambda \in \mathbb{F}_{q^2}$ is a non-zero element with $\lambda + \lambda^q = 0$. Then, in any case, $\rho(t) = t^{-1}$. Thus, $\langle t, \rho \rangle$ is a dihedral subgroup of $\mathrm{Aut}(G)$ of twice odd order. In particular, t is a power of $\rho^t \rho = t^2$, and hence $t \in \langle \rho, \rho^t \rangle$. Our claim follows from Lemma 2.3 in case q is even. If q is odd, we let $^-$ denote the natural homomorphism $G \to G/Z(G)$, and $\bar{\rho}$ the automorphism of $\bar{G}$ induced by ρ. Then $\bar{H} = C_{\bar{G}}(\bar{\rho})$. By Lemma 2.3, we may conclude that $\bar{t} \in C_{\bar{G}}(\bar{H} \cap \bar{H}^{\bar{t}})$. As $\bar{t}$ has order prime to $q+1$, our assertion follows from Corollary 2.7. $\diamond$

TABLE 5. Bounds for $\mathcal{C}_5$-type groups; $q = q_0^k$

G	bc_G^{-1}	$\tilde{H}$	h
$\mathrm{SL}_n(q)$	$q^{n(n-1)/2}(q-1)^{n-1}$	$\mathrm{GL}_n(q_0).(q-1)$	$q_0^{n^2}(q-1)$
$\mathrm{SU}_n(q)$ $n \geq 3$	$\frac{q^2-q+1}{q^2-1} q^{n(n-1)/2}(q^2-1)^{[n/2]}(q+1)^{-1}$	$N_{\mathrm{GU}_n(q)}(\mathrm{SU}_n(q_0))$ $\mathrm{GO}_n^{\epsilon}(q).(q+1)$ $\mathrm{Sp}_n(q).(q+1)$	$q_0^{n^2-1}(q+1)^2$ $2q^{n(n-1)/2}(q+1)$ $q^{n(n+1)/2}(q+1)$
$\mathrm{Sp}_n(q)$ $n \geq 4$	$q^{n^2/4}(q-1)^{n/2}$	$\mathrm{Sp}_n(q_0).2$	$2q_0^{n(n+1)/2}$
$\Omega_n(q)$ $2 \nmid qn$	$\frac{1}{2} q^{(n-1)^2/4}(q-1)^{(n-1)/2}$	$\mathrm{GO}_n(q_0)$	$2q_0^{n(n-1)/2}$
$\Omega_n^+(q)$ $n \geq 8$	$\frac{1}{2} q^{n(n-2)/4}(q-1)^{n/2}$	$\mathrm{GO}_n^-(q^{1/2})$ $\mathrm{GO}_n^+(q_0)$	$2q^{n(n-1)/4}$ $2q_0^{n(n-1)/2}$
$\Omega_n^-(q)$ $n \geq 8$	$\frac{1}{2} q^{n(n-2)/4}(q-1)^{(n-2)/2}(q+1)$	$\mathrm{GO}_n^-(q_0)$	$2q_0^{n(n-1)/2}$

We summarize the results of this subsection.

PROPOSITION 6.24. *Let G be one of the following groups.*

(a) *A group as in Hypothesis 6.3.*

(b) *A group $G = \mathrm{Spin}_n^{\epsilon}(q)$, $n \geq 7$, with q odd, $\epsilon \in \{-1, 0, 1\}$ and $(n, q) \neq (7, 3)$.*

Let H be a non-parabolic maximal subgroup of G of $\mathcal{C}_5$-type. Then H is not the stabilizer of an imprimitivity decomposition of an irreducible KG-module.

PROOF. The result follows from Lemmas 6.22 and 6.23. $\diamond$

6.2.6. The case H is of type $\mathcal{C}_6$. Here we consider maximal subgroups H of G such that $H = N_G(R)$, where R is a group of extraspecial type (we use the notation and terminology of [**69**, §4.6]). In particular, R is an r-group for some prime r, and R is the largest normal r-subgroup of H.

LEMMA 6.25. *Let G be one of the groups of Hypothesis 6.3 or $G = \Omega_6^+(q)$. If H is a maximal $\mathcal{C}_6$-type subgroup of G, then $|H|c_G/|B| < 1$.*

PROOF. Let R be an r-group of extraspecial type such that $H = N_G(R)$.

If either r is odd, or $Z(R)$ is of order 4, then $H/RZ(G) \cong \mathrm{Sp}_{2m}(r)$ and $G = \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$ with $n = r^m$ (see [**69**, Table 4.6.B, p. 150]). If $r = 2$, then $r \leq q$ and $c_G = 1$ as n is even. If r is odd, then $r \mid q-1$ or $r \mid q+1$ and $c_G \leq (q^2-1)/(q^2-q+1) \leq q$. Thus, unless $r = q+1$ is a Fermat prime, we have $|H|c_G \leq q^{2+2m}q^{2m^2+m}|Z(G)|$. Also, $|B| \geq q^{n(n-1)/2}|Z(G)|$. Considering exponents we see that $|H|c_G/|B| \leq 1$ whenever

$$2m^2 + 3m + 2 \leq r^m(r^m-1)/2.$$

The latter is true unless $m = 1$ and $r = 2, 3$, or $m = 2 = r$ or $m = 3$ and $r = 2$. Using sharper estimates for $|B|$ and $|H|$, we find that $|H|c_G/|B| < 1$ in these cases as well. The case that $r = q+1$ is a Fermat prime is ruled out using similar considerations.

If $Z(R)$ is of order 2, then $r = 2$ and H/R is isomorphic to a subgroup of $\mathrm{GO}_{2m}^{\pm}(2)$ with $m \geq 1$, and $G = \mathrm{Sp}_n(q)$ or $\Omega_n^+(q)$ with $n = 2^m$ and q odd (see [**69**, Table 4.6.B, p. 150]). If $m = 1$, then $G = \mathrm{Sp}_2(q)$ with q odd, and H is the normalizer of a quaternion group of order 8. As $|H| \leq 48$ and $q \geq 11$ by assumption, we have $|H| < q(q-1) = |B|$. If $m = 2$, then $G = \mathrm{Sp}_4(q)$ with q odd (the case $G = \Omega_4^+(q)$ is excluded by our assumptions on G), and $H \leq 2^{1+4}\mathrm{GO}_4^-(2)$. Thus $|B| = q^4(q-1)^2$ and $|H| \leq 32\cdot 72$. As $q > 3$, it follows that $|H|/|B| < 1$. If $m \geq 3$, then $G = \mathrm{Sp}_{2^m}(q)$ and $H \leq 2^{1+2m}\mathrm{GO}_{2m}^-(2)$ or $G = \Omega_{2^m}^+(q)$ and $H \leq 2^{1+2m}\mathrm{GO}_{2m}^+(2)$. Using our standard bounds for $|H|$ and $|B|$, we find

$$\frac{|H|}{|B|} \leq \frac{2^{2m^2+m+1}}{(q-1)^{2^{m-1}}q^{2^{m-1}(2^{m-1}-1)}}.$$

The latter is clearly smaller than 1 if $q \geq 5$ or if $q = 3$ and $m \geq 4$. The case $q = 3$ and $m = 3$ can be settled using better estimates for $|H|$. $\diamond$

We summarize the results of this subsection.

PROPOSITION 6.26. *Let G be one of the following groups.*

(a) *A group as in Hypothesis 6.3.*

(b) *A group* $G = \mathrm{Spin}_n^{\epsilon}(q)$, $n \geq 7$, *with* q *odd,* $\epsilon \in \{-1, 0, 1\}$ *and* $(n, q) \neq (7, 3)$, *or* $G = \mathrm{Spin}_6^{+}(q)$.

Let H *be a non-parabolic maximal subgroup of* G *of* $\mathcal{C}_6$*-type. Then* H *is not the stabilizer of an imprimitivity decomposition of an irreducible* KG*-module.*

PROOF. The result follows from Lemma 6.25. $\diamond$

6.2.7. The case H is of type $\mathcal{C}_8$. Suppose that H is a subgroup of G of type $\mathcal{C}_8$. Table 6 displays the possible pairs (G, H) that need to be considered. Notice that the case $G = \mathrm{Sp}_n(q)$ and q even has already been treated in the subsection on $\mathcal{C}_1$-type subgroups.

LEMMA 6.27. *Let* $G = \mathrm{SL}_n(q)$ *with* $n \geq 3$ *and* $(n, q) \neq (3, 4)$. *If* H *is a maximal subgroup of orthogonal or unitary* $\mathcal{C}_8$*-type, then* $h/b \leq 1$.

PROOF. This is clear from the estimates in Table 6. Notice that $q \geq 4$ if H is unitary. $\diamond$

LEMMA 6.28. *If* $G = \mathrm{SL}_n(q)$ *with* $n \geq 3$ *and* H *is a maximal subgroup of symplectic* $\mathcal{C}_8$*-type, then there exists an element* $t \in G \setminus H$ *with* $t \in C_G(H \cap H^t)$.

PROOF. Let $G = \mathrm{SL}_n(q)$ with $n = 2m \geq 4$ even. By σ we denote the automorphism of G sending A to $\tilde{J}^{-1}A^{-T}\tilde{J}$. Then $C := C_G(\sigma) = \mathrm{Sp}_{2m}(q)$. We have $N_{\mathrm{GL}_n(q)}(C) = \langle C, s_\lambda \rangle$ with

$$s_\lambda := \begin{pmatrix} \lambda I & 0 \\ 0 & I \end{pmatrix},$$

where λ is a generator of $\mathbb{F}_q^*$. Let $d := \gcd(m, q-1)$ and put $s := s_\lambda^{(q-1)/d}$. Then s has order d and determinant 1, and $H = N_G(C) = \langle C, s \rangle$.

If q is even and $m = 2, 4$, or if $q = 2$, we let

$$t = \begin{pmatrix} A & 0 \\ 0 & JA^TJ \end{pmatrix},$$

with

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & I_{m-2} \end{pmatrix},$$

where $A_0 \in \mathrm{SL}_2(q)$ has order $q+1$. In all other cases we put

$$t = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

where A is chosen as follows. If q is even (and hence $m \neq 2, 4$), we let A be a diagonal matrix of determinant 1 and order $q-1$ such that the eigenspaces of A for different eigenvalues have distinct dimension, and such that $J^{-1}A^{-T}J = A^{-1}$. If q is odd, we put

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & I_{m-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\sigma(t) = t^{-1}$ and thus $\langle t, \sigma \rangle$ is a dihedral subgroup of $\mathrm{Aut}(G)$ of twice odd order. In particular, t is a power of $\sigma^t \sigma = t^2$, and hence $t \in \langle \sigma, \sigma^t \rangle$. As t commutes with s, the first two hypotheses of Lemma 2.3 are satisfied.

Clearly, $\langle t \rangle$ is a characteristic subgroup of $\langle z, z^t \rangle$. In the even q case we have $N_G(\langle t \rangle) = C_G(t)$ by our conditions on the eigenvalues. Thus t centralizes $H \cap H^t$ by Lemma 2.4.

Suppose now that q is odd. In this case we verify Condition (3) of Lemma 2.3. Let $X \in C$ such that $t^{-1}Xt = Ys^i$ for some $Y \in C$ and some $0 < i < d$. Thus

$$t^{-1}Xts^{-i} \in C.$$

Now $\sigma(t) = t^{-1}$ and $\sigma(s) = \nu^{-1}s$ for some $\nu \in \mathbb{F}_q^*$ of order d, and so

$$t^{-1}Xts^{-i} = \sigma(t^{-1}Xts^{-i}) = \nu^i tXt^{-1}s^{-i}.$$

It follows that

$$t^2 X \nu^i = Xt^2.$$

As $\nu^i \neq 1$, this implies that $X = 0$, a contradiction. $\diamond$

TABLE 6. Bounds for $\mathcal{C}_8$-type groups

G	b	$\tilde{H}$	h
$\mathrm{SL}_n(q)$	$q^{n(n-1)/2}(q-1)^{n-1}$	$\mathrm{Sp}_n(q).d$	$q^{n(n+1)/2}$
n odd		$\mathrm{GO}_n(q)$	$q^{n(n-1)/2}$
		$\mathrm{GO}_n^{\pm}(q)$	$q^{n(n-1)/2}$
		$\mathrm{GU}_n(q^{1/2})$	$(q^{1/2}+1)q^{(n^2-1)/2}$
$\mathrm{Sp}_n(q)$	$q^{n^2/4}(q-1)^{n/2}$	$\mathrm{GO}_n^{\pm}(q)$	$q^{n(n-1)/2}$
$n \geq 4$, and q even			

We summarize the results of this subsection.

PROPOSITION 6.29. *Let G be one of the groups of Hypothesis 6.3(a) or (c).*

Let H be a non-parabolic maximal subgroup of G of $\mathcal{C}_8$-type. Then H is not the stabilizer of an imprimitivity decomposition of an irreducible KG-module.

PROOF. The result follows from Lemmas 6.27, 6.28 and 6.8. $\diamond$

6.2.8. The case H is of type $\mathcal{S}$. In this subsection we let G be one of the groups of Hypothesis 6.3, except that we drop the restrictions on q for the linear, unitary and symplectic groups, but still requiring that G be quasisimple. We also allow for $G = \Omega_7(3)$ and $G = \Omega_8^+(2)$. In [**69**, p. 3], Kleidman and Liebeck give a definition of the class $\mathcal{S}$ of subgroups of the simple group $\bar{G} := G/Z(G)$. If $\hat{G}$ is a quasisimple central extension of $\bar{G}$, we define the $\mathcal{S}$-type subgroups of $\hat{G}$ to be the inverse images of the $\mathcal{S}$-type subgroups of $\bar{G}$. The definition of Kleidman and Liebeck implies in particular, that subgroups of G of type $\mathcal{S}$ are not of type $\mathcal{C}_8$.

Suppose that H is a maximal subgroup of G from class $\mathcal{S}$. Then $Z(H) = Z(G)$ and we write $\bar{H} := H/Z(G)$. Also, $\bar{H}$ is almost simple and thus $F^*(\bar{H})$ is a non-abelian finite simple group. Finally, $F^*(H)$ is quasisimple and acts irreducibly on V.

Recall that V is the natural vector space of G and that B denotes a Borel subgroup of G. In this situation the main theorem of Liebeck [**73**, Theorem 4.1] states that either $|\bar{H}| \leq |V|^3$ or $F^*(\bar{H})$ is an alternating group and V is the reduced permutation module for $F^*(\bar{H})$. We consider each possibility in turn.

In the first case we easily obtain the following estimate.

LEMMA 6.30. *If $|\bar{H}| \leq |V|^3$, then $|H|c_G/|B| < 1$ unless G is linear and* $\dim(V) \leq 7$*, or G is unitary or symplectic and* $\dim(V) \leq 12$*, or G is orthogonal and* $\dim(V) \leq 14$*.*

PROOF. This is proved using $|\bar{H}| \leq q^{3n}$, respectively $|\bar{H}| \leq q^{6n}$ if G is unitary, together with the lower bounds for $|B|$ given in Table 1, for example. $\diamond$

Now the tables in Hiss-Malle [**52**] and Lübeck [**79**] leave only the following list of possibilities.

LEMMA 6.31. *If H is a maximal subgroup of $\mathcal{S}$-type (but not of $\mathcal{C}_8$-type) and either G is linear and $dim(V) \leq 7$, or G is unitary or symplectic and $dim(V) \leq 12$, or G is orthogonal and $dim(V) \leq 14$, then one of the following is true.*

(1) *If $F^*(\bar{H})$ is a simple group of Lie type, cross characteristically embedded into G, or if $F^*(\bar{H})$ is a sporadic group, then $F^*(\bar{H})$ is one of*

the following groups: $\mathrm{PSL}_2(q)$ *with* $q \leq 29$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSp}_4(3)$, $\mathrm{PSp}_4(5)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSp}_6(3)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(4)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_5(2)$, $\Omega_8^+(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$, M_i *where* $i \in \{11, 12, 22, 23, 24\}$, J_1, J_2, J_3, Suz.

(2) *If* $F^*(\bar{H})$ *is a simple group of Lie type of characteristic equal to that of* G*, then* $F^*(\bar{H})$ *is of type* A_1*, of type* A_2 *or* ${}^2A_2$ *and* $\dim(V) \in \{6, 7, 8, 10\}$*, of type* A_3 *or* ${}^2A_3$ *and* $\dim(V) \in \{10, 14\}$*, of type* A_4 *or* ${}^2A_4$ *and* $\dim(V) = 10$*, of type* B_2 *and* $\dim(V) \in \{10, 12, 13, 14\}$*, of type* ${}^2B_2$ *and* $\dim(V) = 4$*, of type* B_3 *and* $\dim(V) \in \{8, 14\}$*, of type* C_3 *and* $\dim(V) \in \{8, 13, 14\}$*, or of type* G_2 *or* ${}^2G_2(q)$ *and* $\dim(V) \in \{6, 7, 14\}$.

(3) *If* $F^*(\bar{H})$ *is alternating and* V *is not the reduced permutation module, then* $F^*(H) = A_n$ *or* $2.A_n$ *with* $n \leq 8$*, or* $F^*(H) = 2.A_9$ *or* $2.A_{10}$ *and* $\dim(V) = 8$.

To deal with the case where G and $F^*(H)$ are of Lie type of equal characteristic, we need the following lemma, whose formulation and proof is due to Frank Lübeck. For the notation used see [**79**, Section 2].

LEMMA 6.32. *Let* p *be a prime and* $\mathbb{F}$ *an algebraic closure of the finite field* $\mathbb{F}_p$ *with* p *elements.*

Let $\mathbf{H}$ *be a simple, simply connected algebraic group over* $\mathbb{F}$*, defined over* $\mathbb{F}_p$*, and let* F *denote the corresponding standard Frobenius map. Furthermore, let*

$$\Psi : \mathbf{H} \to \mathrm{GL}_n(\mathbb{F})$$

be an irreducible rational representation of $\mathbf{H}$ *with highest weight* λ.

Let r *be a positive integer,* $q := p^r$*, and put* $H := \mathbf{H}^{F^r}$*. Assume that* λ *is* q*-restricted such that* $\mathrm{Res}_H^{\mathbf{H}}(\Psi)$ *is irreducible by Steinberg's theorems* [**104**, Theorems 7.4, 9.2]. *Let* $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^{r-1}\lambda_{r-1}$ *be the* p*-adic decomposition of* λ *(i.e., the* λ_i *are* p*-restricted,* $0 \leq i \leq r-1$*).*

Let s *be a positive integer and write* i' *for the non-negative remainder of* $s + i$ *modulo* r *for all* $0 \leq i \leq r-1$*. Suppose that* $\Psi(H) \leq \mathrm{GL}_n(p^s)$*. Then* $\lambda_i = \lambda_{i'}$ *for all* $0 \leq i \leq r-1$

In particular, $s \geq r$*, if* $\lambda = p^i\lambda_i$ *for some* $0 \leq i \leq r-1$.

PROOF. By Steinberg's tensor product theorem [**104**, Theorem 1.1], the representation Ψ is equivalent to a representation on the twisted tensor product

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_{r-1})^{(r-1)},$$

where the tensor products are taken over $\mathbb{F}$. The fact that $\Psi(H) \leq \mathrm{GL}_n(p^s)$ implies that $\mathrm{Res}_H^{\mathbf{H}}(L(\lambda))$ is isomorphic to $\mathrm{Res}_H^{\mathbf{H}}(L(\lambda)^{(s)})$. Now $L(\lambda)^{(s)} \cong L(\lambda_0)^{(s)} \otimes L(\lambda_1)^{(s+1)} \otimes \cdots \otimes L(\lambda_{r-1})^{(s+r-1)}$. The restriction

of the latter module to H is isomorphic to the restriction to H of $L(\lambda_0)^{(0')} \otimes L(\lambda_1)^{(1')} \otimes \cdots \otimes L(\lambda_{r-1})^{((r-1)')}$. The assertion follows from the uniqueness result of Steinberg's theorems [**104**, Theorems 7.4, 9.2]. $\diamond$

The examples that still need closer inspection are collected in the next lemma.

LEMMA 6.33. *Let H be a maximal subgroup of G of $\mathcal{S}$-type. If $|\bar{H}| \leq |V|^3$ and $\dim(V) \leq 14$, then $|H|c_G/|B| < 1$ unless one of the following holds.*

(1) *The group $F^*(\bar{H})$ is of Lie type, cross characteristically embedded into G, or $F^*(\bar{H})$ is a sporadic group, and (G, H) is as in the following table.*

G	H
$\mathrm{SL}_3(4)$	$3 \times \mathrm{PSL}_2(7)$
$\mathrm{SU}_4(3)$	$4_2.\mathrm{PSL}_3(4)$
$\mathrm{Sp}_6(2)$	$\mathrm{PSU}_3(3)$
$\mathrm{SU}_6(2)$	$3.\mathrm{PSU}_4(3)$
$\mathrm{SU}_6(2)$	$3.M_{22}$
$\Omega_7(3)$	$\mathrm{Sp}_6(2)$
$\Omega_8^+(3)$	$2.\Omega_8^+(2)$

(2) *The group $F^*(\bar{H})$ is of Lie type of characteristic equal to that of G and (G, H) is one of $(\mathrm{Sp}_6(q), G_2(q))$, $(\Omega_8^+(q), \mathrm{Sp}_6(q))$ with q even, or one of $(\Omega_8^+(q), \mathrm{Spin}_7(q))$, $(\Omega_7(q), G_2(q))$ with q odd.*

(3) *The group $F^*(\bar{H})$ is alternating, V is not the reduced permutation module and (G, H) is as in the following table.*

G	H
$\mathrm{SL}_2(9)$	$2.A_5$
$\mathrm{SL}_2(11)$	$2.A_5$
$\mathrm{SL}_3(4)$	$3.A_6$
$\mathrm{SU}_3(5)$	$3.A_7$
$\mathrm{Sp}_4(2)$	A_6
$\mathrm{SU}_4(2)$	A_6
$\Omega_8^+(2)$	A_9

PROOF. For each simple group S from the lists in Lemma 6.31, we consider the quasisimple classical groups G possibly containing a maximal subgroup H of type $\mathcal{S}$ with $F^*(\bar{H}) = S$. We then use the precise values for the orders of $\mathrm{Aut}(S)$ and sharp bounds for the orders of the Borel subgroups of G to exclude further possibilities.

The first and third part are straightforward computations. For the second part we use Lemma 6.32 to bound the order of the underlying field of G from below. $\diamond$

The second alternative of Liebeck's result is treated in the following lemma.

LEMMA 6.34. *If $F^*(\bar{H}) = A_m$ and V is the reduced permutation module then $|H|c_G/|B| < 1$ unless $q = 2$ and $n \leq 14$ or $q = 3$ and $n \leq 7$.*

PROOF. By definition, $n \in \{m-1, m-2\}$, and V carries a non-degenerate quadratic or symplectic form. In particular, $c_G = 1$. Writing $\bar{B} := B/Z(G)$ we thus have $|\bar{B}| \geq q^{n(n-2)/4}$. Also, $|\bar{H}| \leq m! \leq (n+2)!$ unless $m = 6$. Notice that $|\bar{H}|/|\bar{B}| = |H|/|B|$.

Put
$$f(n,q) := \frac{(n+2)!}{q^{n(n-2)/4}}.$$
Then
$$f(n+1,q)/f(n,q) = \frac{n+3}{q^{(2n-1)/4}} < 1$$
for all $q \geq 2$ and all $n \geq 8$. Thus $f(n,q)$ is increasing for fixed $q \geq 2$ and $n \geq 8$.

We have $f(8,4) < 1$, hence $f(n,q) < 1$ for all $n \geq 8$ and $q \geq 4$. Using better bounds for $|\bar{H}|$ and $|\bar{B}|$ we show that $|H| < |B|$ for all $q \geq 4$ and all $n \geq 3 \leq 7$.

We have $f(10,3) < 1$, and hence $f(n,3) < 1$ for all $n \geq 10$. The case $q = 3$ and $n = 8$ cannot occur, and for $q = 3$ and $n = 9$ we use sharper bounds for $|\bar{H}|$ and $|\bar{B}|$ to obtain our result.

We finally have $f(16,2) < 1$ and thus $f(n,2) < 1$ for all $n \geq 16$. The case $q = 2$ and $n = 15$ cannot occur. $\diamond$

LEMMA 6.35. *Let G be one of the groups of Hypothesis 6.3. Suppose that (G,H) occurs in the conclusion of Lemma 6.33, but*
$$(G,H) \notin \{(\mathrm{SL}_2(11), 2.A_5), (\mathrm{SU}_3(5), 3.A_7), (\mathrm{Sp}_6(q), G_2(q)),\ q \textit{ even}\}.$$
If $G = \Omega_7(q)$ or $\Omega_8^+(q)$ with q odd, we write $\hat{G}$ for the corresponding spin group and $\hat{H}$ for the inverse image of H in $\hat{G}$.

Then there is $t \in G \setminus H$ respectively $\hat{t} \in \hat{G} \setminus \hat{H}$ such that $t \in C_G(H \cap H^t)$, respectively $\hat{t} \in C_{\hat{G}}(\hat{H} \cap \hat{H}^{\hat{t}})$.

PROOF. If (G,H) occurs in conclusion (1) or (3) of Lemma 6.33, then either G is excluded by Hypothesis 6.3 or
$$(G,H) \in \{(\Omega_8^+(3), 2.\Omega_8^+(2)), (\mathrm{SL}_2(11), 2.A_5), (\mathrm{SU}_3(5), 3.A_7)\}.$$

Suppose that $(G, H) = \{(\Omega_8^+(3), 2.\Omega_8^+(2))$. The following results are easily obtained by a computation in GAP in the permutation representation of $\bar{G}$ on the cosets of $\bar{H}$. The number $|H\backslash G/H|$ of double cosets of H in G equals 6. The largest double coset is of size 22 400. The corresponding point stabilizer $H \cap H^x$ contains a Sylow 3-subgroup of H. No other double coset point stabilizer contains a Sylow 3-subgroup of H. The centralizer $C_G(H \cap H^x)$ is cyclic of order 6 and contains an element t of order 3 which does not lie in H. Thus $H \neq H \cap H^t \geq H \cap H^x$. So in fact $H \cap H^t = H \cap H^x$, as HxH is the unique double coset whose point stabilizer contains a full Sylow 3-subgroup of H. Thus $t \in C_G(H \cap H^t)$ which is our claim. The corresponding result for $\mathrm{Spin}_8^+(3)$ follows from the above in connection with Corollary 2.7

If (G, H) occurs in conclusion (2) and if $H \neq G_2(q)$, then $G = \Omega_8^+(q)$ and $H = \mathrm{Spin}_7(q)$ ($\cong \mathrm{Sp}_6(q)$ if q is even). There is a triality automorphism τ of $\bar{G} = P\Omega_8^+(q)$ sending $\bar{H}$ to a $\mathcal{C}_1^*$-type subgroup $\bar{H}^\tau$ of $\bar{G}$ (see [**66**, Proposition 2.2.4]). Write H_1 for the inverse image of $\bar{H}^\tau$ in G. By Lemmas 6.5 and 6.6, there is an element $t_1 \in G \setminus H_1$ such that t_1 centralizes $H_1 \cap H_1^{t_1}$. Moreover, if q is odd, t_1 has odd order. Write $\bar{t}_1$ for the image of t_1 in $\bar{G}$. Then $\bar{t}_1 \notin \bar{H}^\tau$ and $\bar{t}_1$ centralizes $\bar{H}^\tau \cap (\bar{H}^\tau)^{\bar{t}_1}$.

Put $\bar{t} := \bar{t}_1^{\tau^{-1}}$. Then $\bar{t} \in \bar{G} \setminus \bar{H}$ and $\bar{t} \in C_{\bar{G}}(\bar{H} \cap \bar{H}^{\bar{t}})$. If q is even, $\bar{G} = G$, and we are done. If q is odd, $\bar{t}$ has odd order, and the result for (G, H) and for $(\hat{G}, \hat{H})$ follows from Corollary 2.7.

Suppose now that $(G, H) = (\Omega_7(q), G_2(q))$ and q is odd. The group $H \cong G_2(q)$ contains maximal subgroups $H^\epsilon \cong \mathrm{SL}_3^\epsilon(q)$, $\epsilon \in \{1, -1\}$, such that H^ϵ stabilizes a vector $v_\epsilon \in V$ whose stabilizer in $G \cong \Omega_7(q)$ equals $\mathrm{SO}_6^\epsilon(q)$ (see [**2**]). Choose $\epsilon \in \{1, -1\}$ such that $q \equiv \epsilon \bmod 4$. Then, by [**106**, p. 168], G contains an involution t which fixes v_ϵ and acts as $-I$ on $\langle v_\epsilon \rangle^\perp$. Now t is not in H, as t centralizes $H^\epsilon = C_H(v_\epsilon)$. Thus $C_H(v_\epsilon)$ is contained in $H \cap H^t \neq H$. So by the maximality of $C_H(v_\epsilon)$ in H we obtain equality and so t centralizes $H \cap H^t$.

The element t lifts to an element $\hat{t}$ of order 4 in $\mathrm{Spin}_7(q)$. In fact, $\hat{t}$ generates the center of $\mathrm{Spin}_6^\epsilon(q) \cong \mathrm{SL}_4^\epsilon(q)$, hence commutes with the inverse image of H^ϵ. Thus our claim also holds for $\hat{G} = \mathrm{Spin}_7(q)$. This completes our proof. $\diamond$

LEMMA 6.36. *Suppose that G is one of the groups described in Hypothesis 6.3 and that $(\bar{G}, \bar{H})$ occurs in the conclusion of Lemma 6.34. Then G does not have an irreducible imprimitive cross characteristic representation with block stabilizer H.*

PROOF. If $(\bar{G}, \bar{H})$ occurs in the conclusion of Lemma 6.34, then $F^*(\bar{H})$ is an alternating group, and V is the reduced permutation module. If $q = 3$, then $n \leq 7$. If $n = 7$, then $\bar{G} \cong \Omega_7(3)$, a case excluded by Hypothesis 6.3. If $n = 6$, then $\bar{G} \cong P\Omega_6^-(3) \cong \mathrm{PSU}_4(3)$, again excluded by Hypothesis 6.3. The case $n = 5$ does not occur. If $n = 4$, then $\bar{H} \cong A_6$ and $\bar{G} \cong P\Omega_4^-(3) \cong A_6$.

If $q = 2$ then n is even and $4 \leq n \leq 14$. Let V denote the truncated permutation module for A_{n+2}. Then V restricts to the truncated permutation module for A_{n+1}. Also, V carries a non-degenerate, A_{n+2}-invariant quadratic form, if and only if $n = \dim(V)$ is not divisible by 4 (see [**69**, p. 187] or [**43**, p. 637]). In this case, this form is also S_{n+2}-invariant, but the elements of $S_{n+2} \setminus A_{n+2}$ do not lie in $\Omega(V)$. If $4 \mid n$, then we get an embedding of A_{n+1} as a maximal subgroup of $\Omega_n^\epsilon(2)$ for $\epsilon \in \{+, -\}$, and an embedding of S_{n+2} as a maximal subgroup of $\mathrm{Sp}_n(2) = \mathrm{PSp}_n(2)$.

For $n = 4$ we obtain embeddings of A_5 into $\Omega_4^-(2) \cong \mathrm{PSL}_2(4) \cong A_5$ and of A_6 into $\mathrm{PSp}_4(2)$, so this case need not be considered. The case $n = 6$ leads to $\bar{G} \cong \Omega_6^+(2) \cong \mathrm{PSL}_4(2) \cong A_8$, so again we are done. If $n = 8$, we get an embedding of A_9 into $\Omega_8^+(2)$ and of S_{10} into $\mathrm{PSp}_8(2)$. The former case is excluded by Hypothesis 6.3. In the latter case we use the known ordinary and modular character tables of $\bar{G} = \mathrm{PSp}_8(2)$ to show that no irreducible KH-module induces to an irreducible KG-module. If $n = 10$ we obtain an embedding of A_{12} into $\Omega_{10}^-(2)$. All character tables of $\Omega_{10}^-(2)$ are available in GAP (except for the 2-modular table, which is not relevant in this case). It is then easy to check that $[\bar{G} : \bar{H}]$ does not divide the degree of any irreducible cross characteristic representation of $\Omega_{10}^-(2)$. Next, let $n = 12$. Suppose that $H = A_{13}$, embedded in the 12-dimensional orthogonal group $G = \Omega_{12}^\epsilon(2)$. As $[13!/2^{31}] = 2$, it follows that only 1-dimensional KH-modules can possibly induce to irreducible KG-modules. As $H = A_{13}$ is perfect, the only 1-dimensional KH-module induces to a reducible KG-module. We also have an embedding of $H = S_{14}$ as a maximal subgroup of $\mathrm{Sp}_{12}(2)$. The ordinary character table of $\mathrm{Sp}_{12}(2)$ is available in GAP, so that it is trivial to check that the largest dimension of an irreducible KG-module is smaller than the index of S_{14} in $\mathrm{Sp}_{12}(2)$. Finally, suppose that $n = 14$. Then A_{16} embeds as a maximal subgroup into $\Omega_{14}^\epsilon(2)$. As $[16!/2^{43}] = 2$, we are done as in the case $n = 12$. Our proof is complete now. ◇

LEMMA 6.37. *If q is even and $(G, H) = (\mathrm{Sp}_6(q), G_2(q))$, then G does not have an irreducible imprimitive KG-module with block stabilizer H.*

PROOF. If $q = 2$, then G has an exceptional Schur multiplier, and the claim follows from our results in Chapter 5. Thus let $q > 2$. According to our assumption at the beginning of this chapter, $\mathrm{char}(K)$ is odd, as q is even. Then the smallest dimension of a non-trivial irreducible KH-module equals $q^3 - 1$ (see [**46**, Section 9.2, p. 126]). We have

$$\begin{aligned} \frac{[G:H](q^3-1)}{[G:B]} &= \frac{q^3(q^4-1)(q^3-1)}{(q+1)^3(q^2+1)(q^4+q^2+1)} \\ &= \frac{1}{q^4(q-3)-q-1} > 1 \end{aligned}$$

for $q > 4$. Hence, by Lemma 6.2, no nontrivial KH-module induces to an irreducible KG-module. Clearly, the trivial KH-module does not induce to an irreducible module, and our claim follows. $\diamond$

We finally rule out the remaining possibilities.

LEMMA 6.38. *If* $(G, H) \in \{(\mathrm{SL}_2(11), 2.A_5), (\mathrm{SU}_3(5), 3.A_7)\}$*, then* G *does not have an irreducible imprimitive* KG*-module with block stabilizer* H*.*

PROOF. If $(G, H) = (\mathrm{SU}_3(5), 3.A_7)$, then $[\mathrm{SU}_3(5) : 3.A_7)] = 50$ but G does not have irreducible ordinary or ℓ-modular characters whose degree is divisible by 50 (see [**65**]), hence the conclusion. If $(G, H) = (\mathrm{SL}_2(11), 2.A_5)$, then $[G : H] = 11$ and G has a unique irreducible character of degree 11 for $\ell \neq 2, 3$, and no irreducible character of degree divisible by 11 for $\ell = 2, 3$ (again see [**65**]). However H is perfect and again we get the desired conclusion. $\diamond$

We summarize the results of this subsection.

PROPOSITION 6.39. *Let* G *be one of the following groups.*

(a) *A group as in Hypothesis 6.3.*

(b) *A group* $G = \mathrm{Spin}_n^\epsilon(q)$*,* $n \geq 7$*, with* q *odd,* $\epsilon \in \{-1, 0, 1\}$ *and* $(n, q) \neq (7, 3)$.

Let H *be a non-parabolic maximal subgroup of* G *of* $\mathcal{S}$*-type. Then* H *is not the stabilizer of an imprimitivity decomposition of an irreducible* KG*-module.*

PROOF. This follows from the above lemmas. $\diamond$

6.3. The exceptional groups of Lie type

Here, we prove Theorem 6.1 for quasisimple groups G of exceptional Lie type, where we exclude the groups already considered in Chapter 5, i.e.,

$$G \notin \{{}^2B_2(8), {}^2G_2(3)', G_2(2)', G_2(3), G_2(4), {}^2F_4(2)', F_4(2), {}^2E_6(2)\}.$$

The Suzuki group ${}^2B_2(2)$ is excluded as it is not quasisimple. Throughout, H denotes a non-parabolic maximal subgroup of G.

6.3.1. The Case $G_2(q)$. Let $G = G_2(q)$, $q > 4$. The maximal subgroups of G have been classified, see Aschbacher [**2**], Kleidman [**67**], Cooperstein [**22**] and Migliore [**89**]. The ordinary character degrees of G are also known (see the web page [**80**] by Frank Lübeck). By inspection we see that the only possibilities for our block stabilizer H are $\mathrm{SL}_3(q).2$ and $\mathrm{SU}_3(q).2$. Moreover, inspection of the character degrees of G and H shows that if M_1 is an irreducible KH-module such that $\mathrm{Ind}_H^G(M_1)$ is irreducible, then M_1 is 1-dimensional, and hence $\dim_K(\mathrm{Ind}_H^G(M_1)) = [G : H]$. If $\mathrm{char}(K) = 0$, we use the table of character degrees of G to show that there is no irreducible character χ of G with $\chi(1) = \frac{1}{2}\, q^3\,(q+1)\,(q^2-q+1)$ or $\frac{1}{2}\, q^3\,(q-1)\,(q^2+q+1)$, i.e., such an M_1 does not exist. By Lemma 2.11, this implies that no such M_1 exists for K of any characteristic.

6.3.2. The Case $F_4(q)$. Let $G = F_4(q)$, $q > 2$. The non-parabolic subgroups H whose index is smaller than the largest character degree of G are $\mathrm{Spin}_9(q)$, $N_G(\mathrm{Spin}_8^+(q))$, and ${}^3D_4(q).3$ (see Liebeck-Saxl [**74**] in connection with Lemma 6.2).

We first consider the case $G = F_4(q)$, q odd, and $H = \mathrm{Spin}_9(q)$. Here $H = C_G(a)$ for an involution $a \in G$ whose trace on the 26-dimensional $\mathbb{F}_qG$-module N is -6. It is well known that G contains exactly two classes of involutions which are distinguished by their traces on N. It is also well known that G contains a subgroup isomorphic to $\mathrm{SO}_3(q) \times G_2(q)$ whose action on N is $(N_3 \otimes N_7) \oplus N_5$, where N_3 and N_7 are the natural modules for $\mathrm{SO}_3(q)$ and $G_2(q)$, respectively, and N_5 is the $\mathrm{SO}_3(q)$-module of bivariate homogeneous polynomials of degree 4 (and on which $G_2(q)$ acts trivially). For these facts see [**3**]. Thus the trace of an involution from $\Omega_3(q)$ on N is -6, and hence it is conjugate to a. Thus every involution in $\Omega_3(q)$ is conjugate to a. Now $\Omega_3(q) \cong \mathrm{PSL}_2(q)$ contains two involutions whose product has odd order $\neq 1$. By Lemma 2.3 (with $s = 1$), an irreducible KH-module will never induce to an irreducible KG-module.

In case q even and $H = \mathrm{Spin}_9(q) = \mathrm{Sp}_8(q)$, there does not exist an element t such that $t \in C_G(H^t \cap H)$; thus we argue as follows. The index of H in G equals $q^8(q^8 + q^4 + 1)$.

Suppose that $\mathrm{char}(K)$ is 0 or odd, and let M_1 be an irreducible KH-module with $M := \mathrm{Ind}_H^G(M_1)$ irreducible. Since H is perfect, the dimension of M_1 is larger than 1. Also, $\dim(M_1) < (q^4-1)(q^3-1)q^2$, since otherwise $\dim(M) > [G\colon B]$, where B denotes a Borel subgroup

of G, contradicting the upper bound of Seitz given in Lemma 6.2. Hence, by the result [**45**, Theorem 1.1] of Guralnick and Tiep, M_1 is a Weil module.

Suppose now that $\mathrm{char}(K) = 0$. The degrees of the ordinary Weil characters of H are given in [**45**, Table 1] as polynomials (over the rationals) in q. The degrees of the ordinary irreducible characters of G can be computed from Lusztig's Jordan decomposition of characters. These degrees, again as polynomials in q, are explicitly given on the web site [**80**] of Frank Lübeck. Using these data together with Lemma 2.13, one can easily show that no ordinary irreducible character χ of G satisfies $\chi(1) = [G\colon H]\psi(1)$ for some ordinary Weil character ψ of H.

Hence $\mathrm{char}(K) = \ell \neq 0$. Let φ denote the Brauer character of M_1. By [**45**, Table 1], there is an ordinary Weil character ψ of H such that φ occurs in the reduction modulo ℓ of ψ and such that $\varphi(1) = \psi(1)$ or $\varphi(1) = \psi(1) - 1$. If $\varphi(1) = \psi(1)$, the character φ is liftable to ψ, and thus $\mathrm{Ind}_H^G(\varphi)$ is reducible by Lemma 2.11 and by the above result in the $\mathrm{char}(K) = 0$ case. Thus $\varphi(1) = \psi(1) - 1$.

Since induction commutes with reduction modulo ℓ, there is an irreducible constituent χ of $\mathrm{Ind}_H^G(\psi)$ such that the irreducible Brauer character $\mathrm{Ind}_H^G(\varphi)$ of M is a constituent in the reduction modulo ℓ of χ. Hence

$$[G\colon H](\psi(1) - 1) \leq \mathrm{Ind}_H^G(\varphi)(1) \leq \chi(1) \leq [G\colon H]\psi(1). \tag{6.2}$$

Using Lemma 2.13 once more, one can show that no ordinary irreducible character χ of G satisfies (6.2). This contradiction proves that H is not a block stabilizer of some imprimitive irreducible KG-module.

Next, let $H = N_G(\mathrm{Spin}_8^+(q)) = \mathrm{Spin}_8^+(q).S_3$. Using the result of Seitz given in Lemma 6.2, together with the bounds on minimal degrees by Landazuri and Seitz [**71**] for $\mathrm{Spin}_8^+(q)$, we find that if M_1 is an irreducible KH-module such that $\mathrm{Ind}_H^G(M_1)$ is irreducible, then $\mathrm{Spin}_8^+(q)$ is in the kernel of M_1, i.e., M_1 may be viewed as a KS_3-module and $\dim_K(M_1) \leq 2$. Since S_3 is solvable we may assume that $\mathrm{char}(K) = 0$ in order to rule out H as a block stabilizer (cf. Lemma 2.11). If q is not divisible by 2 or 3, then q^{12} divides $[G\colon H]$. In any case q^{11} divides this index. Using Frank Lübeck's explicit list of character degrees of G (see [**80**]), we find that no degree of an ordinary irreducible character of G is divisible by $[G\colon H]$.

Finally, suppose that $H = {}^3D_4(q).3$. We choose a subgroup $Z := G_2(q)$ of H. We note that Z is unique up to conjugacy in H and that $C_H(Z)Z$ is the unique maximal overgroup of Z in H (see [**68**]). Also, $C_H(Z) = \langle z\rangle$ is cyclic of order 3. Moreover, $C_G(Z) = \mathrm{SO}_3(q)$ (see [**99**]).

Since $q > 2$, we can choose $1 \neq t \in C_G(Z)$ such that $C_H(Z) \cap C_H(Z)^t = 1$. We claim that $H \cap H^t = Z$. If not, $H \cap H^t = Z \times \langle z \rangle$, and thus $z \in H^t$. It follows that $z \in C_{H^t}(Z) = \langle z \rangle^t$, a contradiction. We conclude that t centralizes $H \cap H^t$ and thus, as $t \notin H$, no irreducible KG-module is induced from a KH-module.

6.3.3. The Case $E_6(q)$ or ${}^2E_6(q)$. Let $G = d.E_6(q)$ with $d = \gcd(3, q-1)$ or $G = d.{}^2E_6(q)$ with $d = \gcd(3, q+1)$ and $q \neq 2$. Again, by Lemma 6.2 and the results of Liebeck and Saxl [**74**], the non-parabolic subgroups H of G, whose index is smaller than the largest character degree of G are $Z(G)F_4(q)$ and, in case $G = d.{}^2E_6(q)$, also $N_G(\mathrm{Spin}_{10}^-(q))$. (To rule out the particular case $G = E_6(2)$ and $H = \mathrm{SL}_2(2) \times \mathrm{SL}_6(2)$ we use the ordinary character degrees of G available in GAP [**37**].)

Let $H = Z(G)Z$ with $Z \cong F_4(q)$. We may assume that $Z = C_G(\sigma)$ for the graph automorphism σ of G (see [**42**, Table 4.5.2 and Proposition 4.9.2(b)(4)]). There is a σ-invariant Levi subgroup L of G of type $A_5(q)$ or ${}^2A_5(q)$, respectively, and σ acts as a graph automorphism on L. Next, there is a non-trivial element $t \in L$ of order prime to 6, which is inverted by σ; for example we can choose t to be a non-trivial element of order prime to 6 of a σ-invariant Singer cycle of L. Now write $\bar{\ }\colon G \to G/Z(G)$ for the canonical epimorphism. As $\bar{t} \notin \bar{H}$, Lemma 2.3 (with $s = 1$) shows that no irreducible $K\bar{H}$-module can induce to an irreducible $K\bar{G}$-module. By Corollary 2.7, the same conclusion holds for H.

Now let $G = d.{}^2E_6(q)$ and $H = N_G(\mathrm{Spin}_{10}^-(q))$. If q is odd, $H = C_G(a)$, where a is an involution of the subgroup $F_4(q) \leq G$ with centralizer $\mathrm{Spin}_9(q)$. In Subsection 6.3.2 we have constructed two conjugates of a in $F_4(q)$ whose product has odd order $\neq 1$. Thus we are done with Lemma 2.3 (with $s = 1$).

If q is even, $H = \mathrm{Spin}_{10}^-(q) \times (q+1)$ (see [**76**, Table 5.1]). Using Lemma 6.2, together with the Landazuri and Seitz bounds [**71**] for $\mathrm{Spin}_{10}^-(q)$, we find that if M_1 is an irreducible KH-module inducing to an irreducible KG-module, then $\dim_K(M_1) = 1$. Now use the table of the irreducible character degrees of G provided on Frank Lübeck's web page (see [**80**]), to prove that $[G\colon H]$ does not divide any such degree if $\mathrm{char}(K) = 0$. The same conclusion then holds for any characteristic of K by Corollary 2.7.

6.3.4. The Case $E_8(q)$. Let $G = E_8(q)$. The non-parabolic subgroups H whose index is smaller than the largest character degree of G are $H = (\mathrm{SL}_2(q) \circ E_7(q)).2$ if q is odd, and $H = \mathrm{SL}_2(q) \times E_7(q)$ if q is even (see Liebeck-Saxl [**74**] in connection with Lemma 6.2). The first factor S of $F^*(H)$ is a long root subgroup of $E_8(q)$ isomorphic

to $\mathrm{SL}_2(q)$. Now let L be a long root subgroup of $E_8(q)$ isomorphic to $\mathrm{SL}_3(q)$ containing S. It is known (see e.g., [**42**, Table 4.7.3A]) that $N_G(L) = L \times E_6(q)$ if $q \not\equiv 1 (\mathrm{mod}\ 3)$ and $N_G(L) = (L \circ E_6(q)) \colon \langle s \rangle$, otherwise, where s induces a diagonal automorphism of order 3 on each factor. Clearly L respectively $\langle L, s\rangle$ is isomorphic to a subgroup of $\mathrm{GL}_3(q)$, so Lemma 2.10 guarantees the existence of an element $t \in L$ such that t centralizes $N_L(S) \cap N_L(S)^t$ respectively $N_{\langle L,s\rangle}(S) \cap N_{\langle L,s\rangle}(S)^t$. Thus the hypotheses of Lemma 2.5 are satisfied (with $s = 1$ if $q \not\equiv 1(\mathrm{mod}\ 3)$), showing that t centralizes $H \cap H^t$. Thus H is not the block stabilizer of an imprimitivity decomposition of any irreducible KG-module.

6.3.5. The Case $E_7(q)$. Let $G = d.E_7(q)$ with $d = \gcd(2, q-1)$. To simplify the exposition, we put $E_6^+(q) := \gcd(3, q-1).E_6(q)$, and $E_6^-(q) := \gcd(3, q+1).{}^2E_6(q)$. Using once more the result of Liebeck and Saxl [**74**] together with Lemma 6.2, we find that the non-parabolic subgroups H whose index is smaller than the largest character degree of G are $H = N_G(E_6^\epsilon(q))$, $\epsilon \in \{+,-\}$, and $H = N_G(S)$ where $S \cong \mathrm{SL}_2(q)$ is a long root subgroup of G. If $H = N_G(S)$, then the argument from Subsection 6.3.4 goes through verbatim with one minor difference. Namely that now $C_G(L)$ is $A_5(q)$ rather than $E_6(q)$.

Let $H = N_G(E_6^\epsilon(q))$ and put $e := \gcd(3, q-\epsilon)$. Then $H = (S \circ E_6^\epsilon(q)).e.\langle\sigma\rangle = N_G(S)$, where S is a cyclic torus of order $q - \epsilon$, and σ induces the graph automorphism on $F^*(H) = E_6^\epsilon(q)$ (see [**74**, Table 1]). As already observed in Subsection 6.3.3, we have $Z := C_{E_6^\epsilon(q)}(\sigma) \cong F_4(q)$. Let $\bar{\ }\colon G \to G/Z(G)$ denote the canonical epimorphism. We identify Z with its image in $\bar{G}$. Then $N_{\bar{G}}(Z) = L \times Z = (L' \times Z).\langle\sigma\rangle = N_{\bar{G}}(L')$ with $\mathrm{PSL}_2(q) \cong L' \leq L \cong \mathrm{PGL}_2(q)$. For $q > 3$ this follows from [**77**, Theorem 2(e)]. For $q = 2, 3$, we may apply the same theorem to the groups $d.E_7(q^2)$ and consider the fixed points of the field automorphism. As $\bar{S}$ centralizes Z, we have $\bar{S} \leq L'$. Now suppose that $q > 3$. Then Lemma 2.9 guarantees the existence of $t \in L'$ with $L' = \langle \bar{S}, \bar{S}^t\rangle$ and t centralizing $N_L(\bar{S}) \cap N_L(\bar{S})^t$. It follows from Lemma 2.5 that t centralizes $\bar{H} \cap \bar{H}^t$, and so no irreducible $K\bar{H}$-module induces to an irreducible $K\bar{G}$-module. As we may choose such a t of odd order if q is odd, the same conclusion holds for H and G by Corollary 2.7.

Now suppose that $q = 3$. Then $H = (E_6(3) \times 2).2$ or $H = ({}^2E_6(3) \times 4).2$. Let $x \in C_G(E_6^\epsilon(3))$ have order 2, if $\epsilon = +$, and order 4, otherwise. With this notation we have $H = ({}^2E_6(3) \times \langle x\rangle).\langle\sigma\rangle$, and the group generated by x and σ is elementary abelian of order 4, respectively dihedral of order 8. Moreover, $C_H(Z) = \langle x, \sigma\rangle$, and $Z \times C_H(Z)$ is the unique maximal subgroup of H containing Z but not $F^*(H) = E_6^\epsilon(3)$.

We have $N_G(Z) = L \times Z$ with $L \cong \mathrm{GL}_2(3)$. Let $t \in L$ be an element of order 3. As $t \notin H = N_G(F^*(H))$, the intersection $H \cap H^t$ does not contain $F^*(H)$. From the preceding remark we have

$$H \cap H^t \leq Z \times L = N_G(Z)$$

and

$$H \cap N_G(Z) = Z \times \langle x, \sigma\rangle.$$

This implies

$$\begin{aligned} H \cap H^t &= H \cap H^t \cap N_G(Z) \\ &= [H \cap N_G(Z)] \cap [H^t \cap N_G(Z)] \\ &= Z \times [\langle x, \sigma\rangle \cap \langle x, \sigma\rangle^t]. \end{aligned}$$

By our choice of t we know that t centralizes Z. A direct computation in $L \cong \mathrm{GL}_2(3)$ shows that the intersection of any two distinct dihedral subgroups of order 8 has order 2, i.e., lies in the center of L. The same conclusion holds for any two distinct elementary abelian subgroups of order 4 of L. It follows that $t \in C_G(H \cap H^t)$.

Finally, suppose that $q = 2$. Then $G = E_7(2)$ and $H = E_6(2){:}\,2$ or $H = (3.{}^2E_6(2)){:}\,S_3$ (see [**74**, Table 1] or [**21**, p. 219]). The case $H = E_6(2){:}\,2$ is setteled as follows. We have $N_G(Z) = Z \times L$ with $L \cong \mathrm{GL}_2(2)$, and $N_H(Z) = Z \times \langle\sigma\rangle$. Let $t \in L$ be an element of order 3. Then $t \notin H$, and thus $H \cap H^t = Z$. As $t \in C_G(Z)$, we are done. Now let $H = (3.{}^2E_6(2)){:}\,S_3$. Then $H = N_G(\langle z\rangle)$, where z is an element of order 3 in the center of $F^*(H) = 3.{}^2E_6(2)$. By the tables of conjugacy classes of elements of order 3 given on Frank Lübeck's web page (see [**80**]), z is the unique element of order 3, up to conjugacy, whose centralizer contains an element of order 13. Now G contains a subgroup D isomorphic to ${}^3D_4(2)$ centralizing a subgroup L isomorphic to $\mathrm{SL}_2(8)$ (see [**76**, Table 5.1] or [**21**, p. 219]). As D contains an element of order 13, we may assume that $z \in L$. We have $N_G(L) = (L \times D).\langle s\rangle$ with s an element of order 3 normalizing L and D and acting non-trivially on each factor. In particular, $\langle L, s\rangle \cong \mathrm{Aut}(\mathrm{SL}_2(8)) \cong {}^2G_2(3)$. A computation inside ${}^2G_2(3)$ shows that there exists an element $t \in L$ such that $L = \langle z, z^t\rangle$ and such that t centralizes $N_{\langle L,s\rangle}(\langle z\rangle) \cap N_{\langle L,s\rangle}(\langle z\rangle)^t$. Lemma 2.5 implies that t centralizes $H \cap H^t$ and thus H is not a block stabilizer in G.

6.3.6. The remaining cases. First we consider the Steinberg triality groups ${}^3D_4(q)$, the Ree groups ${}^2F_4(2^{2m+1})$, $m \geq 1$, and ${}^2G_2(3^{2m+1})$, $m \geq 1$ and the Suzuki groups ${}^2B_2(2^{2m+1})$, $m \geq 1$. (Recall that the Tits group ${}^2F_4(2)'$ has alreay been considered in Proposition 5.1.) The maximal subgroups of the Steinberg triality groups were determined

by Kleidman [**68**], those of the large Ree groups by Malle [**88**], those of the small Ree groups by Kleidman [**67**]) and those of the Suzuki groups by Suzuki [**105**]. In every case the smallest index of a non-parabolic maximal subgroup exceeds the bound of Seitz given in Lemma 6.2, eliminating these cases.

CHAPTER 7

Groups of Lie type: Induction from parabolic subgroups

Here, we begin the investigation of imprimitive irreducible modules whose block stabilizers are parabolic subgroups. First we show that such an irreducible module is Harish-Chandra induced. Then we give two sufficient conditions for the irreducibility of a Harish-Chandra induced module. Our first condition is in terms of Harish-Chandra theory, and the second one in terms of Deligne-Lusztig theory. The two conditions are sufficient to show that the bulk of all cross-characteristic irreducible representations of groups of Lie type are in fact Harish-Chandra induced, hence imprimitive.

7.1. Harish-Chandra series

In this section we let G be a finite group with a split BN-pair of characteristic p, satisfying the commutator relations (see e.g., [**23**, §§65,69] for the definition and principal results arising from this set of axioms). By a parabolic subgroup, respectively Levi subgroup of G we mean some N-conjugate of a standard parabolic subgroup, respectively standard Levi subgroup of G. As above, we let K denote an algebraically closed field of characteristic $\ell \geq 0$, with $\ell \neq p$.

Proposition 7.1. *Let M be an irreducible KG-module which is imprimitive with block stabilizer P, where P is a parabolic subgroup of G. Then $M = R_L^G(M_1)$ for some KL-module M_1, where L is a Levi complement in P. In other words, M is Harish-Chandra induced.*

Proof. In this proof we write $[X, Y] := \dim_K \mathrm{Hom}_{KH}(X, Y)$ for subgroups $H \leq G$ and KH-modules X and Y.

Let M_2 be an irreducible KP-module such that $\mathrm{Ind}_P^G(M_2) \cong M$. Let Q be the parabolic subgroup opposite to P. Thus $P \cap Q = L$ is a Levi complement of P and of Q. If M_1 is an irreducible constituent of the head of $\mathrm{Res}_L^P(M_2)$, then $[\mathrm{Res}_L^P(M_2), M_1] > 0$. Denote the inflation of M_1 to a KQ-module by $\tilde{M}_1$. Let D be a set of double coset representatives, containing 1, for the P-Q-double cosets of G. By Mackey's

theorem,

$$\begin{aligned}[][\mathrm{Ind}_P^G(M_2), \mathrm{Ind}_Q^G(\tilde{M}_1)] &= [\mathrm{Res}_Q^G \mathrm{Ind}_P^G(M_2), \tilde{M}_1] \\ &= \sum_{x \in D} [\mathrm{Ind}_{P^x \cap Q}^Q \mathrm{Res}_{P^x \cap Q}^{P^x}(M^x{}_2), \tilde{M}_1] \\ &= \sum_{x \in D} [\mathrm{Res}_{P^x \cap Q}^{P^x}(M^x{}_2), \mathrm{Res}_{P^x \cap Q}^Q(\tilde{M}_1)].\end{aligned}$$

Our choice of M_1 implies that the summand corresponding to $x = 1$ in the last line of the formula above is non-zero. It follows that $[\mathrm{Ind}_P^G(M_2), \mathrm{Ind}_Q^G(\tilde{M}_1)] > 0$.

The latter implies that $\mathrm{Ind}_P^G(M_2)$ is isomorphic to a submodule of $\mathrm{Ind}_Q^G(\tilde{M}_1)$, as $\mathrm{Ind}_P^G(M_2)$ is irreducible. On the other hand, our choice of M_1 and the fact that P and Q have the same order, implies that $\dim_K(\mathrm{Ind}_Q^G(\tilde{M}_1)) \leq \dim_K(\mathrm{Ind}_P^G(M_2))$. Thus $\mathrm{Res}_L^P(M_2) = M_1$ and $\mathrm{Ind}_P^G(M_2) \cong \mathrm{Ind}_Q^G(\tilde{M}_1)$. Since $R_L^G(M_1) \cong \mathrm{Ind}_Q^G(\tilde{M}_1)$, the claim follows. $\diamond$

An ordinary irreducible KG-module is called *Harish-Chandra imprimitive*, if it is Harish-Chandra induced from a proper Levi subgroup of G. Otherwise it is called *Harish-Chandra primitive*. By the above proposition, an irreducible KG-module is imprimitive with parabolic block stabilizer, if and only if it is Harish-Chandra imprimitive.

We now apply Harish-Chandra theory to produce Harish-Chandra imprimitive irreducible modules. If L is a Levi subgroup of G we denote Harish-Chandra induction from L to G by R_L^G. Thus if M is a KL-module, then $R_L^G(M) = \mathrm{Ind}_P^G(\tilde{M})$, where P is a parabolic subgroup of G with Levi complement L, and $\tilde{M}$ is the inflation of M to P. It is known that, up to isomorphism, $R_L^G(M)$ is independent of the choice of the parabolic subgroup P with Levi complement L (see [**27, 58**]). If M is irreducible and cuspidal, $W_G(L, M)$ denotes the ramification group of $R_L^G(M)$ (see [**41**, Section 3]).

THEOREM 7.2. *Let L_0 be a Levi subgroup of G and let M_0 be an irreducible cuspidal KL_0-module. Suppose that L is a Levi subgroup of G with $L_0 \leq L$ and $W_G(L_0, M_0) = W_L(L_0, M_0)$. Then $R_L^G(M_1)$ is irreducible for every irreducible quotient or submodule M_1 of $R_{L_0}^L(M_0)$.*

PROOF. The dimension of $\mathrm{End}_{KG}(R_{L_0}^G(M_0))$ equals the order of $W_G(L_0, M_0)$ (see e.g., [**28**, Theorem 2.9]). Moreover, $\mathrm{End}_{KL}(R_{L_0}^L(M_0))$ is embedded into $\mathrm{End}_{KG}(R_{L_0}^G(M_0))$ as a unital subalgebra (see [**44**, 2.5]). Since the two algebras have the same dimension, they are isomorphic.

The (L_0, M_0) Harish-Chandra series of L consists of the set of irreducible submodules of $R_{L_0}^L(M_0)$ (up to isomorphism), and this set is equal to the set of irreducible quotients of $R_{L_0}^L(M_0)$ (see [**39**, 2.2] for more explanations and references). The result now follows from [**41**, Proposition 2.7]. $\diamond$

Remark. The assumption of the theorem is satisfied for every intermediate Levi subgroup $L_0 \leq L \leq G$ if $W_G(L_0, M_0) = \{1\}$.

The following observation—together with the results of later chapters—allows us to determine all irreducible imprimitive modules of groups of Lie type of small rank.

LEMMA 7.3. *Suppose that* $\mathrm{char}(K) = \ell > 0$. *If the irreducible* KL*-modules are liftable for every proper Levi subgroup* L *of* G*, then the irreducible Harish-Chandra imprimitive* KG*-modules are liftable to ordinary Harish-Chandra imprimitive modules of* G*.*

PROOF. The claim is a reformulation of Lemma 2.11(2). $\diamond$

We note that the assumptions of the Lemma above are satisfied for the groups ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, $G_2(q)$, and ${}^3D_4(q)$. The proper Levi subgroups of the listed groups are abelian or of type A_1. By the results of Burkhardt [**18**] and Dipper-James, the irreducible modules (in cross-characteristics) of groups of the latter type are all liftable.

7.2. Lusztig series

Now we give a sufficient condition for Harish-Chandra imprimitivity of an irreducible module of a finite group of Lie type in terms of Deligne-Lusztig theory, more precisely, in terms of Lusztig series of characters. We introduce the relevant notation. Our basic references for the terminology and notation in this section are [**19, 26**]. Let $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure of the prime field $\mathbb{F}_p$ and let F be a Frobenius morphism of $\mathbf{G}$. Let $G = \mathbf{G}^F$ denote the corresponding finite group of Lie type.

Let $\mathbf{G}^*$ denote the group dual to $\mathbf{G}$ in the sense of Deligne and Lusztig (see [**19**, Chapter 4]). By abuse of notation, the Frobenius morphism of $\mathbf{G}^*$ induced by F is also denoted by F. The finite group in duality with G is $G^* := \mathbf{G}^{*F}$. The set of ordinary irreducible characters of G is partitioned into rational Lusztig series $\mathcal{E}(G, [s])$, where $[s]$ ranges over the G^{*F}-conjugacy classes of semisimple elements of G^{*F} (see [**26**, Proposition 14.41]).

An F-stable Levi subgroup of $\mathbf{G}^*$ is called *split*, if it is the Levi complement of an F-stable parabolic subgroup of $\mathbf{G}^*$. The F-fixed

points of split Levi subgroups of $\mathbf{G}^*$ are Levi subgroups of G^* in the sense of groups with split BN-pairs.

Let s be a semisimple ℓ-regular element in G^*. It was shown by Broué and Michel ([**17**], Théorème 2.2), that the set

$$\mathcal{E}_\ell(G,[s]) := \bigcup_{t \in C_{G^*}(s)_\ell} \mathcal{E}(G,[ts]),$$

where $C_{G^*}(s)_\ell$ is the set of ℓ-elements of $C_{G^*}(s)$, is the union of the ordinary irreducible characters in a set of ℓ-blocks of G.

THEOREM 7.4. *Let $s \in G^*$ be semisimple such that $C_{\mathbf{G}^*}(s)$ is contained in a proper split Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$. Let $\mathbf{L}$ be a split Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$.*

Then every ordinary irreducible character of G contained in $\mathcal{E}(G,[s])$ is Harish-Chandra induced from a character of $\mathcal{E}(L,[s])$.

If s is ℓ-regular for some prime ℓ not dividing q, then every irreducible ℓ-modular character of G contained in $\mathcal{E}_\ell(G,[s])$ is Harish-Chandra induced from a Brauer character lying in $\mathcal{E}_\ell(L,[s])$.

PROOF. The Lusztig map $\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}R_L^G$ induces a bijection between $\mathcal{E}(L,[s])$ and $\mathcal{E}(G,[s])$ (see [**26**, Theorem 13.25(ii)] and [**8**, Théorème 11.10]). Since $\mathbf{L}$ is split, the signs $\varepsilon_{\mathbf{G}}$ and $\varepsilon_{\mathbf{L}}$ are equal (see [**19**, Corollary 6.5.7]) and the map R_L^G is just Harish-Chandra induction (see [**26**, p. 81]).

This proves the first part of the theorem. By a result of Broué ([**14**, p. 62, *Remarque*]), Harish-Chandra induction from L to G induces a Morita equivalence between the unions of ℓ-blocks $\mathcal{E}_\ell(L,[s])$ and $\mathcal{E}_\ell(G,[s])$. In particular, irreducible Brauer characters are mapped to irreducible Brauer characters. $\diamond$

7.3. Asymptotics

The above theorem shows that, asymptotically, most of the irreducible characters of a finite group of Lie type in cross-characteristic are Harish-Chandra induced. To make this statement more precise, we have to talk about infinite families of groups, in our case of series of finite groups of Lie type. That is, we fix the type of the group and vary the underlying field. The formal setting for considerations of this type is that of a *generic finite reductive group* as introduced by Broué and Malle in [**15**]; see also [**16**, Section 1.A].

A generic finite reductive group $\mathbb{G} = (\Gamma, F_0)$ consists of a root datum $\Gamma = (X, R, Y, R^\vee)$ and an automorphism F_0 of Γ of finite order. The components X and Y of Γ are free abelian groups of the same finite

rank, the *rank of* $\mathbb{G}$. They are called "character group" and "cocharacter group" of $\mathbb{G}$, respectively. According to the usage in [**15**], we choose notation such that F_0 acts on X (contrary to the usage in [**16**, Section 1.A], where F_0 acts on Y). The Weyl group W of Γ also acts linearly on X. Extending scalars we obtain actions of W and of F_0 on the $\mathbb{R}$-vector space $X_{\mathbb{R}} := X \otimes_{\mathbb{Z}} \mathbb{R}$.

Assume that $\mathbb{G}$ is not equal to one of the particular types described in [**15**, (drc.t2), (drc.t3)] giving rise to the Suzuki and Ree groups. Then any prime number p determines a connected reductive group $\mathbf{G}$ over the algebraic closure of the prime field of characteristic p, together with a maximal torus $\mathbf{T}$, such that Γ is the root datum associated to the pair $(\mathbf{G}, \mathbf{T})$. The Weyl group W of Γ is isomorphic to the Weyl group $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ of $\mathbf{G}$, and the first component X of Γ is isomorphic to the character group of $\mathbf{T}$, the isomorphism being compatible with the actions of $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ and W. Any power $q > 1$ of p determines a Frobenius morphism F of $\mathbf{G}$ such that $\mathbf{T}$ is F-stable. In particular, F acts on the character group of $\mathbf{T}$, inducing the action qF_0 on X. We shall write $\mathbb{G}(q)$ for the finite reductive group $\mathbf{G}^F$ determined in this way by q. The set of groups $\{\mathbb{G}(q) \mid q > 1 \text{ a prime power}\}$ is also called a *series of finite groups of Lie type*. For example, $\{\mathrm{SL}_n(q) \mid q > 1 \text{ a prime power}\}$ (for fixed n) or $\{{}^2E_6(q) \mid q > 1 \text{ a prime power}\}$ are series of finite groups of Lie type.

The Suzuki and Ree groups, excluded in the theorem below, will be dealt with in a later chapter.

LEMMA 7.5. *Let $\mathbb{G} = (\Gamma, F_0)$ be a generic finite reductive group of rank m. Assume that $\mathbb{G}$ is not equal to one of the particular types described in* [**15**, (drc.t2), (drc.t3)]. *We let W denote the Weyl group of $\mathbb{G}$ and write X for the "character group" of Γ, i.e., the first component of the root datum Γ.*

Then there is a bound M (depending only on $\mathbb{G}$) such that for all prime powers q the following holds: If $(\mathbf{G}, F)$ is the pair of a connected reductive group $\mathbf{G}$ and a Frobenius morphism F of $\mathbf{G}$ determined by $\mathbb{G}$ and q, and if $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, then

$$\left|\{s \in \mathbf{T}^F \mid C_{\mathbf{G}}(s)^F \neq \mathbf{T}^F\}\right| \leq M(q+1)^{m-1}.$$

PROOF. Let q be a prime power, and let $(\mathbf{G}, F)$ and $\mathbf{T}$ be as in the statement of the lemma.

An element $s \in T := \mathbf{T}^F$ is *regular*, if its connected centralizer $C_{\mathbf{G}}(s)^\circ$ equals $\mathbf{T}$. By Part (3) of the proof of [**78**, Theorem 2.1], the number of non-regular elements in T is at most equal to $m^2 \cdot 2^m \cdot (q+1)^{m-1}$.

Suppose that $s \in T$ is regular, but that $C_{\mathbf{G}}(s)^F$ strictly contains T. Then, by [**19**, Theorem 3.5.3], there is a non-trivial F-stable $w \in N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} = W$ centralizing s. Thus s is contained in the closed F-stable subgroup $\mathbf{T}' = \{t \in \mathbf{T} \mid \dot{w}^{-1}t\dot{w} = t\}$ of $\mathbf{T}$, where $\dot{w}$ is an element of $N_{\mathbf{G}}(\mathbf{T})$ mapping to w under the natural epimorphism.

To estimate the order of $\mathbf{T}'^F$, we introduce the following notation. If $A \leq X$ is a subgroup of X, we write $\hat{A}$ for the smallest pure subgroup of X containing A. Following [**19**, Section 1.12], we put $A^\perp := \{t \in \mathbf{T} \mid \chi(t) = 1 \text{ for all } \chi \in A\}$. This is a closed subgroup of $\mathbf{T}$ and it is a sub-torus if $A = \hat{A}$. Moreover, for subgroups $A \leq B \leq X$ such that B/A is finite, we have $A^\perp/B^\perp \cong (B/A)_{p'}$, where p is the prime dividing q (see [**9**, Chapter III] for proofs of these facts).

For $v \in W$, let $[v, X]$ denote the image of $v - \mathrm{id}_X$ in X. Then $\mathbf{T}' = [w, X]^\perp$. Since w is F-stable, $[w, X]$ and $\widehat{[w, X]}$ are F-stable as well and $\widehat{[w, X]}^\perp$ is an F-stable sub-torus of $\mathbf{T}'$. Since $w \in W$ is non-trivial, the rank of $[w, X]$, and hence that of $\widehat{[w, X]}$ is smaller than m. By [**78**, Propositon 2.4], the set of F-stable elements of the torus $\widehat{[w, X]}^\perp$ has at most $(q + 1)^{m-1}$ elements.

Let M_0 be the largest order of a group of the form $\widehat{[v, X]}/[v, X]$, where v runs through the elements of W. Then the order of $\mathbf{T}'/\widehat{[w, X]}^\perp$ is at most equal to M_0. Hence $|\mathbf{T}'^F| \leq M_0(q+1)^{m-1}$ and the lemma is proved. $\diamond$

THEOREM 7.6. *Let $\mathbb{G} = (\Gamma, F_0)$, W and X be as in Lemma 7.5. Denote by X' the root lattice of $\mathbb{G}$, i.e., the sublattice of X spanned by the roots R in X. (Note that F_0 and W act on X' and hence on $X'_{\mathbb{R}} := X' \otimes_{\mathbb{Z}} \mathbb{R}$.)*

(a) *Put $W_{\mathrm{split}} := \{w \in W \mid F_0 \circ w \text{ has an eigenvalue } 1 \text{ on } X'_{\mathbb{R}}\}$. Then the number of absolutely irreducible ordinary characters of a group $\mathbb{G}(q)$ in the series $\{\mathbb{G}(q) \mid q > 1 \text{ a prime power}\}$ equals $q^m + O(q^{m-1})$, and*

$$\frac{|W_{\mathrm{split}}|}{|W|}\, q^m + O(q^{m-1})$$

of these characters are Harish-Chandra imprimitive.

(b) *Let ℓ be a prime number and let e be a positive integer dividing $\ell - 1$. If q is a prime power, we write $e(q, \ell) = e$, if ℓ does not divide q and if e equals the order of $q \mod \ell$ in $\mathbb{F}_\ell^*$. Let $W_{(e,\ell)'}$ denote the set of $w \in W$ for which $F_0 \circ w$, acting on $X'_{\mathbb{R}}$, has no eigenvalue of order $\ell^i e$ for some non-negative integer i. Put $W_{\mathrm{split},(e,\ell)'} := W_{\mathrm{split}} \cap W_{(e,\ell)'}$.*

Then the number of absolutely irreducible Harish-Chandra imprimitive ℓ-modular Brauer characters of a group in the set $\{\mathbb{G}(q) \mid q > 1$ *a prime power with* $e(q, \ell) = e\}$ *is at least*

$$\frac{|W_{\mathrm{split},(e,\ell)'}|}{|W|}\, q^m + O(q^{m-1}).$$

PROOF. Choose a prime power q, hence a group $\mathbf{G}$ and a Frobenius morphism F and write $G := \mathbb{G}(q) = \mathbf{G}^F$. Let $\mathbf{G}^*$, F be the dual pair. A *geometric conjugacy class of* $G^* := \mathbf{G}^{*F}$ is the intersection of a conjugacy class of $\mathbf{G}^*$ with G^*, if this intersection is not empty. We have a partition of the set of irreducible characters $\mathrm{Irr}(G)$ of G into Lusztig series,

$$\mathrm{Irr}(G) = \bigcup_{(s)} \mathcal{E}(G, (s)),$$

where (s) runs through the geometric conjugacy classes of semisimple elements of G^* (see [**26**, Proposition 13.17]). For each semisimple element $s \in G^*$, there is a bijection between $\mathcal{E}(G, (s))$ and $\mathcal{E}(C_{G^*}(s), (1))$, the *unipotent characters* of $C_{G^*}(s)$ (see [**26**, Theorem 13.23]). Moreover, the latter are parameterized independently of q. These results are due to Lusztig; for groups $\mathbf{G}$ with a connected center they are contained as a special case in [**83**, 4.23 Main Theorem], and for groups $\mathbf{G}$ with non-connected center, they are proved in [**84**].

Our strategy is to count the geometric conjugacy classes of semisimple elements $s \in G^*$ for which $C_{G^*}(s)$ is a maximal torus of G^*. For such an s, the set $\mathcal{E}(C_{G^*}(s), (1))$ just consists of the trivial character, and hence $\mathcal{E}(G^*, (s))$ contains a unique element. Asymptotically, the elements lying in such Lusztig series account for all irreducible characters of G.

Up to conjugation in G^*, the number of maximal tori in G^* equals the number of F-conjugacy classes of W (see [**19**, Proposition 3.3.3]). It follows from Lemma 7.5, applied to the dual generic reductive group $\mathbb{G}^*$, that the number of geometric conjugacy classes of semisimple elements s of G^* such that $C_{G^*}(s)$ is not a maximal torus of G^* is at most a constant (independent of q) times $(q+1)^{m-1}$. Since the total number of geometric conjugacy classes of semisimple elements of G^* is of the form $q^m + O(q^{m-1})$ (see [**19**, Theorem 3.7.6(i)]), we have proved the first assertion of Part (a) of the theorem.

To prove the second, let $\mathbf{T_0}$ be a maximally split torus of $\mathbf{G}$. Let $\mathbf{T_0}^*$ be the dual torus of $\mathbf{G}^*$. We choose a set of representatives for the F-conjugacy classes of W, and for each representative w an F-stable maximal torus $\mathbf{T}_w^*$ of $\mathbf{G}^*$ obtained from $\mathbf{T_0}^*$ by twisting with w (see

[**19**, p. 85]). By [**19**, Theorem 3.5.3], an F-stable element s in $\mathbf{T}_w^*$ is regular and satisfies $C_{G^*}(s) = {\mathbf{T}_w^*}^F$ if and only if its orbit in $N_{\mathbf{G}^*}(\mathbf{T}_w^*)^F$ has maximal possible length, namely $|N_{\mathbf{G}^*}(\mathbf{T}_w^*)^F/{\mathbf{T}_w^*}^F|$. The latter number equals $|C_{W,F}(w)|$, the order of the F-centralizer $C_{W,F}(w)$ of w (see [**19**, p. 87] and [**19**, Propositions 3.3.6]). Thus $\mathbf{T}_w^*$ gives rise to $([W : C_{W,F}(w)]/|W|)\, q^m + O(q^{m-1})$ conjugacy classes of regular elements of G^* whose centralizer in G^* equals ${\mathbf{T}_w^*}^F$. Note that the number in the numerator of the coefficient at q^m in the above formula equals the number of elements in the F-conjugacy class of W containing w.

To conclude, we have to observe that $\mathbf{T}_w^*$ is contained in a proper split Levi subgroup of $\mathbf{G}^*$ if and only if $F_0 \circ w$ has an eigenvalue 1 on $X'_{\mathbb{R}}$. Now $\mathbf{T}_w^*$ is contained in a proper split Levi subgroup of $\mathbf{G}^*$ if and only if it contains a non-trivial and non-central $\mathbb{F}_q$-split torus of $\mathbf{G}^*$ (see [**15**, Section 3.D]). This is the case if and only if $\mathbf{T}_w^*/Z^\circ(\mathbf{G}^*)$ contains a non-trivial $\mathbb{F}_q$-split torus of $\mathbf{G}^*/Z^\circ(\mathbf{G}^*)$. Here, $Z^\circ(\mathbf{G}^*)$ denotes the connected component of the center of $\mathbf{G}^*$. Let Y' denote the cocharacter group of $\mathbf{G}^*/Z^\circ(\mathbf{G}^*)$. Then $\mathbf{T}_w^*/Z^\circ(\mathbf{G}^*)$ contains a non-trivial $\mathbb{F}_q$-split torus if an only if $F_0 \circ w$, acting on $Y' \otimes_{\mathbb{Z}} \mathbb{R}$ has an eigenvalue 1. But $Y' \otimes_{\mathbb{Z}} \mathbb{R} = X'_{\mathbb{R}}$ (see [**16**, Section 1.A, p. 13]).

To prove Part (b), assume that q satisfies $e(q, \ell) = e$. Let $s \in G^*$ be a semisimple ℓ'-element such that $C_{G*}(s)$ is a maximal torus. By the main result of Broué and Michel [**17**], the (unique) ordinary character in $\mathcal{E}(G^*, (s))$ remains irreducible on reduction modulo ℓ. Moreover, if s' is another such element not conjugate to s in G^*, then the two irreducible ℓ'-modular characters arising from $\mathcal{E}(G^*, (s))$ and $\mathcal{E}(G^*, (s'))$ are distinct. We thus have to count the number of G^*-conjugacy classes of semisimple ℓ'-elements of G^* whose centralizer is a maximal torus.

Let $\mathbf{T}_w^*$ be a maximal torus of $\mathbf{G}^*$ as above. All non-central elements in ${\mathbf{T}_w^*}^F$ are ℓ'-elements if ℓ does not divide the order of $(\mathbf{T}_w^*/Z^\circ(\mathbf{G}^*))^F$. This order is given by evaluating the characteristic polynomial of $F_0 \circ w$, acting on $X'_{\mathbb{R}}$, at q (see [**19**, Proposition 3.3.5]). The characteristic polynomial is a product of cyclotomic polynomials Φ_j. Now $\Phi_j(q)$ is divisible by ℓ if and only if $j = \ell^i e$ for some non-negative integer i (see. e.g., [**60**, Lemma IX.8.1]). This completes the proof. $\diamond$

Let us consider two examples.

Example 7.7. Let q be a power of the prime p, and let $\mathbf{G} = \mathrm{SL}_n(\bar{\mathbb{F}}_p)$, where $\bar{\mathbb{F}}_p$ denotes an algebraic closure of the prime field $\mathbb{F}_p$. The Weyl group W of $\mathbf{G}$ (defined with respect to the maximal torus $\mathbf{T}_0$ of $\mathbf{G}$ consisting of the diagonal matrices) is isomorphic to the symmetric group on n letters and $X_{\mathbb{R}}$ is the truncated permutation module for W. The rank of $\mathbf{G}$ equals $n - 1$.

(a) Suppose first that F is the standard Frobenius morphism of $\mathbf{G}$ (raising every entry of a matrix to its qth power), so that $G = \mathrm{SL}_n(q)$. Then $F_0 = 1$. Now $w \in W$ has eigenvalue 1 on $X_{\mathbb{R}}$, if and only if it has more than one cycle (in its action as a permutation on n points). Hence $|W_{\text{split}}| = n! - (n-1)!$. It follows that for fixed n the proportion of (Harish-Chandra) imprimitive characters of $\mathrm{SL}_n(q)$ tends to $1 - 1/n$ as q tends to infinity.

(b) Now suppose that F is a Frobenius morphism giving rise to the unitary groups, e.g., $F((a_{ij})) = J_n(a_{ij}^q)^{-T}J_n$, for a matrix $(a_{ij}) \in \mathbf{G}$. (For the notation see Subsection 2.1.2.) This F yields $G = \mathbf{G}^F = \mathrm{SU}_n(q)$. Also F_0 acts as $-w_0$ on $X_{\mathbb{R}}$ in this case, where w_0 is the longest element in W. Now $-w_0w$ has eigenvalue 1 on $X_{\mathbb{R}}$, if and only if w_0w has only cycles of odd lengths (in its action as a permutation on n points). Thus $|W_{\text{split}}| = |\{w \in W \mid w \text{ has only cycles of odd lengths}\}| = |\{w \in W \mid w \text{ has odd order}\}|$.

We will give more precise numbers for certain series of groups of low rank in a later chapter.

CHAPTER 8

Groups of Lie type: $\mathrm{char}(K) = 0$

In this chapter we assume that G is a finite group of Lie type (not necessarily quasisimple), which arises from an algebraic group with connected center. We also assume that K is an algebraically closed field of characteristic 0. In this case we can prove a converse to Theorem 7.2, and thus obtain a classification of the irreducible imprimitive KG-modules with parabolic block stabilizers. We begin with some results on Weyl groups.

8.1. Some results on Weyl groups

In the following, W_m denotes a Weyl group of type B_m. The maximal parabolic subgroups of W_m are of the form $W_{m-k} \times S_k$ for $1 \leq k \leq m$. Here, S_k denotes a symmetric group on k letters. The ordinary irreducible characters of W_m are parameterized by bi-partitions of m, and we write χ^α for the irreducible character of W_m labelled by the bi-partition α. Similarly, ζ^β denotes the irreducible character of S_k labelled by the partition β of k. For partitions α, β, and γ of $m-k$, k, and m, respectively, we write $g^\gamma_{\alpha,\beta}$ for the multiplicity of ζ^γ in the induced character $\mathrm{Ind}^{S_m}_{S_{m-k}\times S_k}(\zeta^\alpha \times \zeta^\beta)$.

LEMMA 8.1. *Let $m \geq 2$ and $1 \leq k \leq m$ be positive integers. Moreover, let γ be a partition of k, and let $\beta = (\beta^0, \beta^1)$ be a bi-partition of $m-k$. Then*

$$\mathrm{Ind}^{W_m}_{W_{m-k}\times S_k}(\chi^\beta \times \zeta^\gamma) \tag{8.1}$$

has at least three distinct irreducible constituents. If $k > 1$ and $(m, k) \neq (2, 2), (3, 3)$, then (8.1) *has at least five distinct irreducible constituents.*

PROOF. Suppose first that $k = 1$. The irreducible constituents of $\mathrm{Ind}^{W_m}_{W_{m-1}}(\chi^\beta)$ are of the form χ^α with bi-partitions $\alpha = (\alpha^0, \alpha^1)$ of m satisfying $\alpha^0 = \beta^0$ or $\alpha^1 = \beta^1$. In the first case α^1 is obtained from adding a node to β^1, and in the second case α^0 is obtained from β^0 in this way. (This is a special case of the general formula (8.2) below.) Since $m \geq 2$, at least one of β^0 or β^1 is not the empty partition. If β^0 is not the empty partition, there are at least two distinct partitions

which can be obtained from β^0 by adding a node. The result follows in this case.

Now suppose that $k > 1$. If $(m, k) = (2, 2)$ or $(3, 3)$ we use the CHEVIE [**40**] share package of GAP [**37**] to show that (8.1) has at least three distinct irreducible constituents (in fact in these cases it has at least four). Thus suppose in addition that $(m, k) \neq (2, 2), (3, 3)$. For a bi-partition (α^0, α^1) of m, put $j := |\alpha^0| - |\beta^0|$. Then the multiplicity of $\chi^\beta \times \zeta^\gamma$ in the restriction of χ^α to $W_{m-k} \times S_k$ is equal to 0 unless $0 \leq j \leq k$. In the latter case this multiplicity equals

$$\sum_{\delta^0} \sum_{\delta^1} g^{\alpha^0}_{\beta^0\delta^0} g^{\alpha^1}_{\beta^1\delta^1} g^{\gamma}_{\delta^0\delta^1}, \tag{8.2}$$

where δ^0 and δ^1 run through the partitions of j and $k - j$, respectively. Formula (8.2) follows from [**92**, Theorem 6] and an application of the Littlewood-Richardson rule (see [**64**, 2.8.14]) for symmetric groups. For a detailed deduction of the formula see [**93**, Lemma 3.5.2]).

Let $\delta = (\delta^0, \delta^1)$ be one of the following bi-partitions of k:

$$(\text{-}, \gamma), (\gamma, \text{-}), ((1), \mu), (\mu, (1)), (\nu, \kappa), \tag{8.3}$$

where μ, ν, and κ are partitions of $k - 1$, 2, and $k - 2$, respectively, such that $g^{\gamma}_{(1),\mu} \neq 0$ and $g^{\gamma}_{\nu,\kappa} \neq 0$.

For each (δ^0, δ^1) as above, there is a bi-partition $\alpha = (\alpha^0, \alpha^1)$ of m such that $g^{\alpha^0}_{\beta^0,\delta^0} \neq 0$ and $g^{\alpha^1}_{\beta^1,\delta^1} \neq 0$. By (8.2) this implies that χ^α occurs in (8.1) as a constituent. If $k \geq 4$, then the five δ's of (8.3) give rise to five distinct α's. Indeed, letting β^0 be a partition of m_0, the first component α^0 of α is a partition of m_0, $m_0 + k$, $m_0 + 1$, $m_0 + k - 1$, and $m_0 + 2$, respectively. If $k = 2$ or 3, then $m - k > 0$ by assumption. Thus at least one of β^0 or β^1 is not the empty partition. Assume without loss of generality that β^0 is not empty. Then, by the Littlewood-Richardson rule, there are at least two distinct partitions α^0 of $m_0 + k$ with $g^{\alpha^0}_{\beta^0,\gamma} \neq 0$ and two distinct partitions α^0 of $m_0 + 1$ with $g^{\alpha^0}_{\beta^0,(1)} \neq 0$. For $\alpha^0 = \beta^0$ we also have $g^{\alpha^0}_{\beta^0,(\text{-})} \neq 0$, and we are done. ◇

LEMMA 8.2. *Let W be a finite Weyl group and let W_0 be a proper parabolic subgroup of W (for this notation see e.g.* [**19**, Section 2.3]*). Then $\mathrm{Ind}_{W_0}^W(\psi)$ is reducible for every ordinary irreducible character ψ of W_0. In fact, if W is of type D_m, $m \geq 4$, or of type E_7, then $\mathrm{Ind}_{W_0}^W(\psi)$ has at least three distinct irreducible constituents. If W is of type E_6, then $\mathrm{Ind}_{W_0}^W(\psi)$ has at least four distinct irreducible constituents.*

PROOF. We may assume that W is irreducible and that W_0 is a maximal parabolic subgroup of W. Suppose first that W is not of type

A_m, D_m, or E_6. Then w_0, the longest element in W ([**19**, Proposition 2.2.11]) is contained in the center of W (see [**11**, Planches I–IX]). Since w_0 is not contained in W_0, the result follows from Lemma 2.2. This argument is due to Meinolf Geck. (Of course, the desired result for W of type B_m also follows from Lemma 8.1.)

It remains to consider the cases that W is of type A_m, $m \geq 2$, D_m, $m \geq 4$, and E_6. The latter case is easily checked with CHEVIE. If W is of type A_m, the Littlewood-Richardson rule (see [**64**, 2.8.14]) shows that no irreducible character of W_0 induces to an irreducible character of W.

In case W is of type D_m, $m \geq 4$, we proceed as follows. We write $\tilde{W}_m$ for the Weyl group of type D_m, and embed $\tilde{W}_m$ into a Weyl group W_m of type B_m of index 2 (exactly as in the first of the two embeddings described in [**83**, 4.6]). A maximal parabolic subgroup of $\tilde{W}_m$ is of the form $\tilde{W}_{m-k} \times S_k$ for some $1 \leq k \leq m$. This is embedded into $W_{m-k} \times S_k$ by embedding $\tilde{W}_{m-k}$ into W_{m-k}.

Now let ψ be an irreducible character of $\tilde{W}_{m-k} \times S_k$ inducing to an irreducible character χ of $\tilde{W}_m$. Then $\mathrm{Ind}_{\tilde{W}_m}^{W_m}(\chi)$ has at most two irreducible constituents. This implies that the irreducible constituents of $\mathrm{Ind}_{\tilde{W}_{m-k} \times S_k}^{W_{m-k} \times S_k}(\psi)$ induce to characters of W_m with at most two irreducible constituents. This contradicts Lemma 8.1. $\diamond$

8.2. Harish-Chandra series

Our next aim is to prove a converse of Theorem 7.2 for groups of Lie type arising from an algebraic group with connected center. More specifically, let $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure of $\mathbb{F}_q$ and let F be a Frobenius morphism of $\mathbf{G}$. Let $G = \mathbf{G}^F$ be the corresponding finite group of Lie type. Assume that the center of $\mathbf{G}$ is connected.

THEOREM 8.3. *For G as above, let M be an irreducible imprimitive KG-module whose block stabilizer is a parabolic subgroup of G with Levi complement L. Then there exists a Levi subgroup L_0 of L, an irreducible cuspidal KL_0-module M_0 with $W_G(L_0, M_0) = W_L(L_0, M_0)$, and an irreducible constituent M_1 of $R_{L_0}^L(M_0)$ such that $M \cong R_L^G(M_1)$.*

PROOF. By Proposition 7.1, there is an irreducible KL-module M_1 such that $M \cong R_L^G(M_1)$. Let L_0 be a Levi subgroup of L and M_0 an irreducible cuspidal KL_0-module such that M_1 is a composition factor of $R_{L_0}^L(M_0)$.

By the results of Lusztig in [**83**, 8.3–8.5]), the ramification group $W_G(L_0, M_0)$ is a Weyl group and $W_L(L_0, M_0)$ is a parabolic subgroup

of $W_G(L_0, M_0)$. By the comparison theorem of Howlett and Lehrer [**57**, Theorem 5.9], there are irreducible characters ψ of $W_L(L_0, M_0)$ and χ of $W_G(L_0, M_0)$ corresponding to M_1 and M respectively, such that the multiplicity of M in $R_L^G(M_1)$ equals the multiplicity of χ in the induced character $\mathrm{Ind}_{W_L(L_0,M_0)}^{W_G(L_0,M_0)}(\psi)$. The first of these multiplicities being 1, this is true for the second as well. By Lemma 8.2 this implies that $W_L(L_0, M_0) = W_G(L_0, M_0)$ and we are done. $\diamond$

8.3. Lusztig series

Here we give a characterization of the irreducible Harish-Chandra imprimitive KG-modules in terms of Lusztig series of characters, thus proving a converse to Theorem 7.4 in characteristic 0. Let $\mathbf{G}$, F and $G = \mathbf{G}^F$ be as in Section 8.2 above. In particular, we assume that the center of $\mathbf{G}$ is connected. As in Section 7.2, we let $(\mathbf{G}^*, F)$ be the pair dual to $(\mathbf{G}, F)$, and we write $G^* = \mathbf{G}^{*F}$.

THEOREM 8.4. *Let $s \in G^*$ be semisimple such that $\mathcal{E}(G, [s])$ contains a Harish-Chandra imprimitive element. Then $C_{\mathbf{G}^*}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$.*

PROOF. Let $\chi \in \mathcal{E}(G, [s])$ be Harish-Chandra induced from the Levi subgroup $L = \mathbf{L}^F$ of G. Suppose that χ lies in the (L_0, ψ)-Harish-Chandra series for some Levi subgroup $\mathbf{L}_0^F = L_0 \leq L$ and some cuspidal irreducible character ψ of L_0. We let $\mathbf{L}_0^*$ and $\mathbf{L}^*$ denote split F-stable Levi subgroups of $\mathbf{G}^*$ in duality with $\mathbf{L}_0$ and $\mathbf{L}$ respectively, where we may assume that $s \in \mathbf{L}_0^* \leq \mathbf{L}^*$.

We now use [**83**, (8.5.7) and (8.5.8)]. This implies that $W_G(L_0, \psi)$ is a Coxeter group with canonical generators in bijection to the F-orbits on $S_{C_{\mathbf{G}^*}(s)} \setminus S_{C_{\mathbf{L}_0^*}(s)}$. Here, $S_{C_{\mathbf{G}^*}(s)}$ denotes the set of fundamental reflections of the Weyl group of $C_{\mathbf{G}^*}(s)$. Now $W_G(L_0, \psi) = W_L(L_0, \psi)$ by Theorem 8.3, and hence the number of F-orbits on $S_{C_{\mathbf{G}^*}(s)} \setminus S_{C_{\mathbf{L}^*}(s)}$ is zero. Since $C_{\mathbf{L}^*}(s)$ is a split Levi subgroup of $C_{\mathbf{G}^*}(s)$, it follows that $C_{\mathbf{G}^*}(s) = C_{\mathbf{L}^*}(s)$, i.e., that $C_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$. $\diamond$

CHAPTER 9

Classical groups: char$(K) = 0$

Here we apply the results of the preceding chapters to certain classical groups for which we obtain explicit descriptions of their Harish-Chandra induced irreducible characters. Since we are only dealing with representations over algebraically closed fields of characteristic 0, it seems appropriate to use the language of characters rather than modules.

We begin with a list of groups satisfying the hypotheses of Theorems 8.3 and 8.4.

9.1. The groups

For the remainder of this section we let G be one of the following finite classical groups:

(a) a general linear group $\mathrm{GL}_n(q)$ with $n \geq 2$,
(b) a general unitary group $\mathrm{GU}_n(q)$ with $n \geq 3$,
(c′) a symplectic group $\mathrm{Sp}_n(q)$ with q even and $n \geq 4$ even,
(c) a conformal symplectic group $\mathrm{CSp}_n(q)$ with q odd and $n \geq 4$ even,
(d) a Clifford group $D_n^0(q)$ with q odd and $n \geq 5$ odd,
(e′) a special orthogonal group $\mathrm{SO}_n^{\pm}(q)$ with q even and $n \geq 8$ even,
(e) a conformal special orthogonal group $\mathrm{CSO}_n^{\pm}(q)$ with q odd and $n \geq 8$ even,
(f) a Clifford group $D_n^{\pm,0}(q)$ with q odd and $n \geq 8$ even.

For the definition of these groups we refer to [**19**, Sections 1.19] and [**10**, Chapter IX, § 9, n°5]. The groups in (c) and (d) are dual groups in the sense of Deligne and Lusztig, as well as the groups of types (e) and (f) (see [**82**, 8.1]). As before, we denote the natural (projective) module for G by V.

9.2. Harish-Chandra series

The aim in this section is a characterization of the ordinary irreducible Harish-Chandra imprimitive characters of the above classical groups in terms of Harish-Chandra theory.

Let W denote the Weyl group of G (as finite group with BN-pair). Thus, for example, the Weyl group of $\mathrm{GU}_{2m}(q)$ is the Weyl group of type B_m. The set of fundamental reflections of W is denoted by S. The Weyl groups and root systems of the groups in 9.1 are irreducible of types A, B, C and D.

For every subset $J \subseteq S$ there is a standard Levi subgroup L_J of G, whose Weyl group is the parabolic subgroup $W_J \leq W$, generated by the reflections in J. As before, a Levi subgroup of G is an N-conjugate of some standard Levi subgroup, where N denotes the monomial subgroup of the BN-pair of G. The relative Weyl group $W_G(L)$ of the Levi subgroup L is defined as $W_G(L) := (N_G(L) \cap N)L/L$.

9.2.1. General linear groups. We first investigate the easy case of the general linear group $G = \mathrm{GL}_n(q)$. In this case W may be identified with the subgroup of permutation matrices in G, and S with the subset $\{s_1, \ldots, s_{n-1}\}$, where s_i corresponds to the transposition $(i, i+1)$, $1 \leq i \leq n-1$.

By a composition of n we understand a sequence $(\lambda_1, \ldots, \lambda_r)$ of positive integers summing up to n. Let J be a proper subset of S. By putting $\lambda_0 := 0$, and $\lambda_{j+1} := \min\{k \mid 1 + \sum_{i=1}^{j} \lambda_i \leq k \leq n-1, s_k \notin J\} - \sum_{i=1}^{j} \lambda_i$, for $j \geq 0$ and $\sum_{i=1}^{j} \lambda_i < n$, we inductively associate a composition of n to J. The standard Levi subgroup L_J of G is of the form

$$L_J \cong \mathrm{GL}_{\lambda_1}(q) \times \mathrm{GL}_{\lambda_2}(q) \times \cdots \times \mathrm{GL}_{\lambda_r}(q), \tag{9.1}$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the composition of n associated to J (with the natural diagonal embedding of the right hand side of (9.1) into G).

An irreducible cuspidal character ψ of L_J as above is of the form

$$\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_r, \tag{9.2}$$

where ψ_i is an irreducible cuspidal character of $\mathrm{GL}_{\lambda_i}(q)$ for $1 \leq i \leq r$.

Let us shortly describe $W_G(L_J)$ and $W_G(L_J, \psi)$ for L_J and ψ as above. For every pair (i, j), $1 \leq i < j \leq r$ such that $\lambda_i = \lambda_j$, there is an element $\tau_{ij} \in W_G(L_J)$ swapping $\mathrm{GL}_{\lambda_i}(q)$ with $\mathrm{GL}_{\lambda_j}(q)$ and centralizing the other factors of L_J. In fact, with respect to a suitable basis of V, the natural module for G, the element τ_{ij} is represented by the matrix

$$\begin{pmatrix} I_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\lambda_i} & 0 \\ 0 & 0 & I_b & 0 & 0 \\ 0 & I_{\lambda_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_c \end{pmatrix}, \tag{9.3}$$

with $a = \sum_{k=1}^{i-1} \lambda_k$, $b = \sum_{k=i+1}^{j-1} \lambda_k$, and $c = \sum_{k=j+1}^{r} \lambda_k$ (where I_d denotes the $d \times d$ identity matrix). By [**56**] we have $W_G(L_J) = \langle \tau_{ij} \mid 1 \leq i < j \leq r, \lambda_i = \lambda_j \rangle$. Moreover, $W_G(L_J, \psi) = \langle \tau_{ij} \mid 1 \leq i < j \leq r, \lambda_i = \lambda_j, \psi_i = \psi_j \rangle$. This follows from one of the main results of Chapter 8 of Lusztig's book (namely [**83**, (8.5.13)]).

PROPOSITION 9.1. *Let* $G = \mathrm{GL}_n(q)$, $n \geq 2$, *let* $L_0 = L_J$ *be a proper Levi subgroup of* G *as in* (9.1), *and let* ψ *be a cuspidal irreducible character of* L_0 *as in* (9.2).

If $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ *and* $\psi_1 = \psi_2 = \cdots = \psi_r$, *then every irreducible constituent of* $R_{L_0}^G(\psi)$ *is Harish-Chandra primitive. Otherwise every such constituent is Harish-Chandra imprimitive.*

PROOF. Suppose first that $\lambda_1 = \lambda_2 = \cdots = \lambda_r$ and $\psi_1 = \psi_2 = \cdots = \psi_r$. Then $W_G(L_0, \psi) = W_G(L_0)$ is isomorphic to the symmetric group on r letters. Let L be a Levi subgroup of G such that L_0 is a Levi subgroup of L. Then some $W_L(L_0)$-conjugate L_1 of L is a standard Levi subgroup of G. Obviously, $W_{L_1}(L_0)$ is a proper subgroup of $W_G(L_0)$. In particular, $W_G(L_0, \psi) \neq W_{L_1}(L_0, \psi)$ and so $W_G(L_0, \psi) \neq W_L(L_0, \psi)$. It follows from Theorem 8.3 and Harish-Chandra theory (see [**23**, Theorem (70.15A)]), that no irreducible constituent of $R_{L_0}^G(\psi)$ is Harish-Chandra imprimitive.

Suppose now that $\lambda_i \neq \lambda_j$ for some $1 \leq i, j \leq r$, or that $\lambda_1 = \lambda_2 = \cdots = \lambda_r$, but $\psi_i \neq \psi_j$ for some $1 \leq i, j \leq r$. By conjugating with a suitable element of W we may assume that there is some $i < r$ such that, in the first case, $\lambda_1 = \cdots = \lambda_i$, but $\lambda_j \neq \lambda_1$ for all $j > i$, and, in the second case, $\psi_1 = \cdots = \psi_i$, but $\psi_j \neq \psi_1$ for all $j > i$. (Such a conjugation changes L_0 and ψ, but it does not change $R_{L_0}^G(\psi)$ by [**23**, (70.11)].) Put $k = \sum_{j=1}^{i} \lambda_j$ and $m = n - k$. Let $L \cong \mathrm{GL}_k(q) \times \mathrm{GL}_m(q)$ be the standard Levi subgroup corresponding to the composition (k, m) of n. Then $L_0 \leq L$ and $W_G(L_0, \psi) = W_L(L_0, \psi)$ by the remarks preceding the proposition. It follows from Theorem 7.2 that every constituent of $R_{L_0}^G(\psi)$ is Harish-Chandra imprimitive. ◇

9.2.2. Other classical groups. We now aim at a similar result for the other classical groups. If G is one of the groups of 9.1(b)–(d) or one of the orthogonal groups or Clifford groups of 9.1(e′)–(f) of $+$ type, we put $m := \lfloor n/2 \rfloor$. In the remaining cases we put $m := n/2 - 1$. Then $m \geq 2$. The Weyl group W of G is of type B_m (whereas the root system of W is of type B_m or C_m) or of type D_m, and we write $S = \{s_1, s_2, \ldots, s_m\}$ for the set of fundamental reflections of W. If W is of type B we choose notation so that $s_2, \ldots, s_m$ are conjugate, but s_1 is not conjugate to s_2. If W is of type D we let s_1, s_2, and s_m be the

end nodes of the Dynkin diagram with s_1 and s_2 of distance 2 (if m is larger than 4, this uniquely determines s_1 and s_2).

Let J be a proper subset of S. Let $k := m + 1 - \min\{1 \leq i \leq m \mid s_i \notin J\}$. We call the subset $\{s_1, \ldots, s_{m-k}\}$ of J its component of type B respectively D (in the latter case only if s_1 and s_2 are contained in J). The subset $\{s_{m-k+2}, \ldots, s_m\} \cap J$ of $\{s_{m-k+2}, \ldots, s_m\}$ determines a composition $(\lambda_1, \ldots, \lambda_r)$ of k as above (in case of the general linear group). The standard Levi subgroup L_J of G is of the form

$$L_J \cong G_{n-2k}(q) \times \mathrm{GL}_{\lambda_1}(q^\delta) \times \mathrm{GL}_{\lambda_2}(q^\delta) \times \cdots \times \mathrm{GL}_{\lambda_r}(q^\delta). \tag{9.4}$$

Here, $\delta = 2$ if $G = \mathrm{GU}_n(q)$, and $\delta = 1$, otherwise. If G is not one of the Clifford groups, an explicit description of the embedding of the right hand side of (9.4) as a standard Levi subgroup of G can be found, e.g., in [**49**, 4.4]. For the Clifford groups use duality as in [**83**, 8.5]. The group $G_{n-2k}(q)$ is of the same type as G and corresponds to the component of J of type B or D, respectively; the index $n-2k$ indicates the dimension of the natural module of $G_{n-2k}(q)$. It may happen that the component of type B (respectively D) of J is empty, in which case $n - 2k \in \{0, 1\}$; in this case the group $G_{n-2k}(q)$ is cyclic of order $q - 1$.

Let ψ be an irreducible cuspidal character of $L := L_J$. Then

$$\psi = \psi_0 \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_r, \tag{9.5}$$

where ψ_0 is an irreducible cuspidal character of $G_{n-2k}(q)$, and ψ_i is an irreducible cuspidal character of $\mathrm{GL}_{\lambda_i}(q^\delta)$ for $1 \leq i \leq r$.

Suppose first that s_1 and s_2 are contained in J if W is of type D_m. Then there is, for every $1 \leq i \leq r$, an involution $\sigma_i \in W_G(L)$ which centralizes every direct factor $\mathrm{GL}_{\lambda_j}(q^\delta)$, $1 \leq j \leq r$, of the decomposition (9.4) except $\mathrm{GL}_{\lambda_i}(q^\delta)$. If $G = \mathrm{GU}_n(q)$, the map induced by σ_i on $\mathrm{GL}_{\lambda_i}(q^2)$ is the Frobenius morphism F of Example 7.7(b) (restricted from $\mathrm{GL}_{\lambda_i}(\bar{\mathbb{F}}_q)$ to $\mathrm{GL}_{\lambda_i}(q^2)$). In the other cases σ_i induces (modulo an inner automorphism), the transpose inverse automorphism on $\mathrm{GL}_{\lambda_i}(q)$. With respect to a suitable basis of V (recall that V is the natural module of G), some preimage of σ_i in N is represented by the matrix

$$\begin{pmatrix} I_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\lambda_i} & 0 \\ 0 & 0 & \tilde{I} & 0 & 0 \\ 0 & I_{\lambda_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_a \end{pmatrix}, \tag{9.6}$$

with $a = c = \sum_{j=i+1}^{r} \lambda_j$. Also, $\tilde{I}^2 = I_b$ with $b = n - 2k + 2\sum_{j=1}^{i-1} \lambda_j$. Note that σ_i need not normalize $G_{n-2k}(q)$.

Now suppose that W is of W is of type D_m and either s_1 or s_2 is not contained in J. Without loss of generality we may assume that $s_1 \notin J$. In this case the Levi subgroup L_J is as in (9.4) with $\delta = 1$, $k = m$ and $G_0(q)$ a cyclic group of order $q - 1$. Fix an i, $1 \leq i \leq r$. If λ_i is even, there still is an involution $\sigma_i \in W_G(L)$ as in the general case above. It is represented on V by the matrix (9.6) with identity matrix $\tilde{I}$. If λ_i is odd, then there is, for every $1 \leq j \neq i \leq r$ with λ_j odd, an involution $\sigma_{ij} \in W_G(L)$ inducing the inverse transpose automorphism simultaneously on $\mathrm{GL}_{\lambda_i}(q)$ and on $\mathrm{GL}_{\lambda_j}(q)$, and centralizing the other direct factors of L_J. The matrix of σ_{ij} on V is the product of the matrices for σ_i and σ_j (which do not lie in G, but their product does).

Let us now return to the general case. For every pair (i, j), $1 \leq i < j \leq r$ such that $\lambda_i = \lambda_j$, there is an element $\tau_{ij} \in W_G(L)$ swapping $\mathrm{GL}_{\lambda_i}(q^\delta)$ with $\mathrm{GL}_{\lambda_j}(q^\delta)$ and centralizing the other factors of L. The elements σ_i and τ_{ij} (respectively σ_i, σ_{ij}, and τ_{ij}) generate $W_G(L)$ by the results of Howlett [**56**].

The σ_i and the τ_{ij} (respectively σ_i, σ_{ij}, and τ_{ij}) arise as the non-trivial elements of subgroups of $W_G(L) = (N_G(L) \cap N)L/L$ of the form $(N_{L_1}(L) \cap N)L/L$ for Levi subgroups L_1 of G such that $(N_{L_1}(L) \cap N)L/L$ has order 2. Such Levi subgroups L_1 are groups of F-fixed points of admissible subgroups in the sense of Lusztig [**83**, p. 255]. The σ_i (respectively the σ_i and σ_{ij}) are distinguished from the τ_{ij} by the fact that the former arise from admissible Levi subgroups L_1 for which the rank of the component of the root system of type B respectively D is larger than the corresponding rank for L (type B_1 being distinguished from type A_1 by the length of the root). Note that the σ_i (respectively the σ_i and σ_{ij}) and the τ_{ij} are also distinguished by the fact that the σ_i (respectively the σ_i and σ_{ij}) normalize the direct factors $\mathrm{GL}_{\lambda_1}(q^\delta), \ldots, \mathrm{GL}_{\lambda_r}(q^\delta)$ of L.

Again by [**83**, (8.5.13)]), the group $W_G(L, \psi)$ is generated by the elements σ_i and τ_{ij} (respectively σ_i, σ_{ij}, and τ_{ij}) that leave ψ invariant.

Proposition 9.2. *Let G be one of the classical groups from* 9.1(b)–(f)*, let $L_0 = L_J$ be a proper Levi subgroup of G as in* (9.4)*, and let ψ be a cuspidal irreducible character of L_0 as in* (9.5).

(a) *In case G is an orthogonal group or Clifford group of $+$ type, assume in addition that $n - 2k \geq 4$ (this is equivalent to the assumption that s_1 and s_2 are contained in J).*

Then if $\sigma_i \in W_G(L_0, \psi)$ for all $1 \leq i \leq r$, every irreducible constituent of $R_{L_0}^G(\psi)$ is Harish-Chandra primitive. Otherwise every such constituent is Harish-Chandra imprimitive.

(b) *Let G be an orthogonal group or Clifford group of $+$ type, and assume that $n - 2k = 0$ (this is equivalent to the assumption that s_1 is not contained in J).*

Then if $\sigma_i \in W_G(L_0, \psi)$ for all $1 \leq i \leq r$ with λ_i even, and if for all $1 \leq i \leq r$ with λ_i odd, there is some $1 \leq j \neq i \leq r$ with λ_j odd such that $\sigma_{ij} \in W_G(L_0, \psi)$, every irreducible constituent of $R_{L_0}^G(\psi)$ is Harish-Chandra primitive. Otherwise every such constituent is Harish-Chandra imprimitive.

PROOF. We only prove (a); the proof of (b) is similar. Suppose first that $\sigma_i \in W_G(L_0, \psi)$ for all $1 \leq i \leq r$. Let L be a Levi subgroup of G such that L_0 is a Levi subgroup of L. Then for some i, no preimage of σ_i is contained in L. In particular, $W_G(L_0, \psi) \neq W_L(L_0, \psi)$. It follows as in the proof of Proposition 9.1 that no irreducible constituent of $R_{L_0}^G(\psi)$ is Harish-Chandra imprimitive.

Suppose now that $\sigma_i \notin W_G(L_0, \psi)$ for some $1 \leq i \leq r$. By conjugating with a suitable element of W, we may assume that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_r$, and that $\tau_{ij} \in W_G(L_0, \psi)$ for all $j > i$, but $\tau_{i',i} \notin W_G(L_0, \psi)$ for all $i' < i$ with $\lambda_{i'} = \lambda_i$. Then $\sigma_j \notin W_G(L_0, \psi)$ for all $j > i$, since $\sigma_i = \tau_{ij}\sigma_j\tau_{ij}$. Put $k' = \sum_{j=i}^{r} \lambda_j$, and let $L = G_{n-2k'}(q) \times \mathrm{GL}_{k'}(q^\delta)$. Then $L \neq G$ and $W_G(L_0, \psi) = W_L(L_0, \psi)$. The assertion now follows from Theorem 7.2. $\diamond$

Lets look at an example to see what the conditions in the two propositions really mean.

EXAMPLE 9.3. For $a \in \mathrm{GL}_n(q)$ write $F'(a) := J_n a^{-T} J_n$, where J_n is as in Example 7.7(b).

(a) Let $G = \mathrm{CSO}_{16}^+(q)$ with odd q. Take $J = \{s_2, \ldots, s_5, s_7, s_8\}$. Then $L := L_J = G_0(q) \times \mathrm{GL}_5(q) \times \mathrm{GL}_3(q)$ (so that $r = 2$, $\lambda_1 = 5$ and $\lambda_2 = 3$). We may choose an embedding of the direct product above into G in such a way that (z, a, b) with $z \in \mathbb{F}_q^* = G_0(q)$, $a \in \mathrm{GL}_5(q)$, $b \in \mathrm{GL}_3(q)$ is represented by the matrix

$$\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & zF'(a) & 0 \\ 0 & 0 & 0 & zF'(b) \end{pmatrix}.$$

Thus σ_{12} maps (z, a, b) to $(z, zF'(a), zF'(b))$. Now let ψ_0, ψ_1, and ψ_2 be irreducible cuspidal characters of $G_0(q)$, $\mathrm{GL}_{\lambda_1}(q) = \mathrm{GL}_5(q)$, and $\mathrm{GL}_{\lambda_2}(q) = \mathrm{GL}_3(q)$, respectively. Suppose that the linear characters of the centers of $\mathrm{GL}_5(q)$ and $\mathrm{GL}_3(q)$ induced by ψ_1 and ψ_2 are ξ_1 and ξ_2, respectively.

It is easy to check that $\psi = \psi_0 \otimes \psi_1 \otimes \psi_2$ is invariant under σ_{12}, if and only if $\xi_1 = \xi_2^{-1}$ and $\psi_i(c) = \psi_i(c^{-T})$ for all $c \in \mathrm{GL}_{\lambda_i}(q)$ and $i = 1, 2$.

(b) Next let $G = \mathrm{CSO}_8^+(q)$ with odd q. We wish to determine all irreducible Harish-Chandra imprimitive characters of G corresponding to the Levi subgroup $L_0 = \mathrm{CSO}_4^+(q) \times \mathrm{GL}_2(q)$ (i.e., $L_0 = L_J$ with $J = \{s_1, s_2, s_4\}$). We may choose an embedding of the direct product into G in such a way that (a, b) with $a \in \mathrm{CSO}_4^+(q)$, $b \in \mathrm{GL}_2(q)$ is represented by the matrix

$$\begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \mu_a F'(b) \end{pmatrix},$$

where $\mu_a \in \mathbb{F}_q^*$ is the multiplier of a.

Again, let ψ_0 and ψ_1 be irreducible cuspidal characters of $G_4(q) = \mathrm{CSO}_4^+(q)$ and $\mathrm{GL}_{\lambda_1}(q) = \mathrm{GL}_2(q)$, respectively. We have to determine the exact conditions under which $\psi := \psi_0 \otimes \psi_1$ is invariant under σ_1. Now σ_1 maps (a, b) to $(a, \mu_a F'(b))$. It is easy to check that ψ is invariant under σ_1 if and only if the center of $\mathrm{GL}_2(q)$ is in the kernel of ψ_1 and if $\psi_1(b) = \psi_1(b^{-T})$ for all $b \in \mathrm{GL}_2(q)$.

9.3. Lusztig series

Let $\mathbf{G}$ be a connected reductive algebraic classical group over $\bar{\mathbb{F}}_q$ with connected center, and let F be a Frobenius morphism of $\mathbf{G}$, such that $G = \mathbf{G}^F$ is one of the groups introduced in 9.1. As in Section 7.2, we let $(\mathbf{G}^*, F)$ be the pair dual to $(\mathbf{G}, F)$, and we write $G^* = \mathbf{G}^{*F}$. The natural modules for $\mathbf{G}$ and $\mathbf{G}^*$ are denoted by $\mathbf{V}$ and $\mathbf{V}^*$, respectively.

Let $s \in G^*$ be semisimple. Since the center of $\mathbf{G}$ is connected, the ordinary irreducible characters of G lying in $\mathcal{E}(G, [s])$ can be labelled as $\chi_{s,\lambda}$, where λ is a unipotent character of $C_{G^*}(s)$.

9.3.1. General linear groups. Again we begin with the general linear groups $G = \mathrm{GL}_n(q)$ where we may and will identify $\mathbf{G}$ with $\mathbf{G}^*$. An element of $\mathrm{GL}_n(q)$ is called irreducible if and only if it acts irreducibly on the natural module of $\mathrm{GL}_n(q)$.

PROPOSITION 9.4. *Let $G = \mathrm{GL}_n(q)$. Let $s \in G$ be semisimple, and let λ be a unipotent character of $C_G(s)$. Then the following assertions are equivalent:*

(a) *The irreducible character $\chi_{s,\lambda}$ of G is Harish-Chandra primitive.*

(b) *The minimal polynomial of s is irreducible.*

(c) *There are $m, d \in \mathbb{N}$ with $md = n$ such that s is conjugate in G to a block diagonal matrix with m blocks, all equal to some irreducible matrix.*

(d) *There are $m, d \in \mathbb{N}$ with $md = n$ such that $C_G(s) \cong \mathrm{GL}_m(q^d)$.*

PROOF. Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ with standard Frobenius map F so that $G = \mathbf{G}^F$. Now $\chi_{s,\lambda} \in \mathcal{E}(G, [s])$. Thus $\chi_{s,\lambda}$ is Harish-Chandra imprimitive if and only if $C_{\mathbf{G}}(s)$ is contained in a proper split Levi subgroup of $\mathbf{G}$ (Theorem 7.4). This is the case if and only if $C_{\mathbf{G}}(s)$ fixes a non-trivial proper subspace of $\mathbf{V}$, i.e., if $C_{\mathbf{G}}(s)$ acts reducibly on $\mathbf{V}$.

Let μ denote the minimal polynomial of s, and let μ_1 be an irreducible factor of μ. Then $\mathrm{Ker}(\mu_1(s))$ is invariant under $C_{\mathbf{G}}(s)$. Hence if $\chi_{s,\lambda}$ is Harish-Chandra primitive then $\mu = \mu_1$ is irreducible. Thus (b) follows from (a). The implications from (b) to (c) and from (c) to (d) are clear (with d the degree of μ). Also, if $C_G(s) \cong \mathrm{GL}_m(q^d)$ for $m, d \in \mathbb{N}$ with $md = n$, then a Coxeter torus of $C_G(s)$ is a Coxeter torus of G, and so $C_G(s)$ acts irreducibly on V. It follows that $C_{\mathbf{G}}(s)$ is not contained in a proper split Levi subgroup of $\mathbf{G}$ so that $\chi_{s,\lambda}$ is Harish-Chandra primitive. $\diamond$

Two particular cases are worth noting. The case $d = 1$ corresponds to the central elements $s \in G$, the Harish-Chandra primitive characters arising in this way are the unipotent characters of G (for $s = 1$), and multiples of unipotent characters with linear characters (for $s \neq 1$). The case $d = n$ gives rise to the cuspidal irreducible characters of G.

Note also that Proposition 9.4(d) gives the degrees of the irreducible Harish-Chandra primitive characters of $\mathrm{GL}_n(q)$, via Lusztig's Jordan decomposition of characters.

9.3.2. Other classical groups. We now prove a similar result for the other classical groups. If μ is a monic irreducible polynomial over $\mathbb{F}_{q^2}$, we write μ^* for the monic irreducible polynomial over $\mathbb{F}_{q^2}$ whose roots are the $(-q)$ths powers of the roots of μ, i.e., if $\mu = \prod_{i=1}^{d}(X - \alpha_i)$ with $\alpha_i \in \bar{\mathbb{F}}_q$, then $\mu^* = \prod_{i=1}^{d}(X - \alpha_i^{-q})$. Note that if $\mu \in \mathbb{F}_q[X]$, then μ^* is the monic polynomial whose roots are the inverses of the roots of μ. If G^* is of one of the Clifford groups, i.e., a group as in 9.1 (d) or (f), we have surjective homomorphisms $\mathbf{G}^* \to \bar{\mathbf{G}}^*$, defined over $\mathbb{F}_q$, where $\bar{\mathbf{G}}^* = \mathrm{SO}_n(\bar{\mathbb{F}}_q)$ in Case (d), and $\bar{G}^* = \mathrm{SO}_n^{\pm}(\bar{\mathbb{F}}_q)$ in Case (f). In these cases we write $\bar{s}$ for the image of $s \in G^*$ in $\bar{G}^* := \bar{\mathbf{G}}^{*F}$. In the other cases we let $\bar{\mathbf{G}}^* = \mathbf{G}^*$ and $\bar{s} = s$ for $s \in G^*$. Note that the natural module $\mathbf{V}^*$ of $\mathbf{G}^*$ is also the natural module of $\bar{\mathbf{G}}^*$.

Proposition 9.5. *Let G be one of the classical groups of* 9.1(b)–(f). *Let $s \in G^*$ be semisimple and let λ be a unipotent character of $C_{G^*}(s)$. Then the following conditions are equivalent:*

(a) *The irreducible character $\chi_{s,\lambda}$ of G is Harish-Chandra primitive.*

(b) *Every irreducible factor μ of the minimal polynomial of $\bar{s}$ (acting on the natural module $\mathbf{V}^*$ of $\bar{\mathbf{G}}^*$) satisfies $\mu = \mu^*$.*

(c) *The element s does not lie in any proper Levi subgroup of G^*.*

(d) *If G is a unitary group, there are nonzero $m_i, d_i \in \mathbb{N}$ with $\sum_{i=1}^r m_i d_i = n$ such that $C_G(s) \cong \mathrm{GU}_{m_1}(q^{d_1}) \times \cdots \times \mathrm{GU}_{m_r}(q^{d_r})$.*

If G is one of the other classical groups, there are nonzero $m_i, d_i \in \mathbb{N}$ with $\sum_{i=1}^r m_i d_i \leq n$ such that $C_{-1} \times C_1 \times \mathrm{GU}_{m_1}(q^{d_1}) \times \cdots \times \mathrm{GU}_{m_r}(q^{d_r})$ is isomorphic to a subgroup of $C_{\bar{G}^*}(\bar{s})$ of index at most 2. Here, for $\zeta = -1, 1$, the group C_ζ is defined by $C_\zeta = C_{\bar{\mathbf{G}}^*(\mathbf{V}_\zeta^*)}(\bar{s}_\zeta)^F$, where $\mathbf{V}_\zeta^*$ denotes the ζ-eigenspace of $\bar{s}$ on $\mathbf{V}^*$, $\bar{s}_\zeta$ is the map induced by $\bar{s}$ on $\mathbf{V}_\zeta^*$, and $\bar{\mathbf{G}}^*(\mathbf{V}_\zeta^*)$ is the same type of group as $\bar{\mathbf{G}}^*$, however acting on $\mathbf{V}_\zeta^*$.

Proof. Suppose that μ is a monic irreducible factor of the irreducible polynomial of $\bar{s}$ with $\mu \neq \mu^*$. If $\bar{\mathbf{G}}^* = \mathbf{G}^*$, i.e., if $\mathbf{G}^*$ is not a spin group, then $\mathrm{Ker}_{\mathbf{V}^*}(\mu(\bar{s}))$ is a totally isotropic respectively totally singular subspace of $\mathbf{V}^*$. Since $\mathrm{Ker}_{\mathbf{V}^*}(\mu(\bar{s}))$ is invariant under $C_{\mathbf{G}^*}(s)$, it follows that $C_{\mathbf{G}^*}(s)$ is contained in a proper split Levi subgroup of $\mathbf{G}^*$, and thus (a) does not hold. In case $\mathbf{G}^*$ is a spin group, let $\prod_i \bar{s}_i$ denote the primary decomposition of $\bar{s}$, the $\bar{s}_i$ acting on the subspaces $\mathbf{V}_i^*$ of $\mathbf{V}^*$ (cf. [**36**, (1.10)]). Put $\mathbf{C}_i := C_{\mathrm{SO}(\mathbf{V}_i^*)}(\bar{s}_i)$. By what we have already proved above, there is an i such that $\mathbf{C}_i$ is contained in a proper split Levi subgroup of $\mathrm{SO}(\mathbf{V}_i^*)$. Thus the product $\prod_i \mathbf{C}_i$, embedded in the natural way into $\bar{\mathbf{G}}^*$, is contained in a proper split Levi subgroup of $\bar{\mathbf{G}}^*$. By [**36**, **(2B)**], the centralizer $C_{\mathbf{G}^*}(s)$ is the inverse image of $\prod_i \mathbf{C}_i$ under the surjection from $\mathbf{G}^*$ to $\bar{\mathbf{G}}^*$. It follows that $C_{\mathbf{G}^*}(s)$ is contained in a proper split Levi subgroup of $\mathbf{G}^*$. Hence (a) implies (b).

If (a) does not hold then $C_{\mathbf{G}^*}(s)$ is contained in a proper split Levi subgroup of $\mathbf{G}^*$. This implies that $\bar{s}$ is contained in a proper Levi subgroup of $\bar{\mathbf{G}}^*$, and thus fixes a non-trivial totally isotropic, respectively totally singular subspace of $\mathbf{V}^*$. The restriction of $\bar{s}$ to this subspace has a minimal polynomial all of whose irreducible factors μ satisfy $\mu \neq \mu^*$. Thus (b) implies (a).

The equivalence of (b) and (d) follows from the description of the centralizers of semisimple elements in the classical groups as given in [**35**, Proposition (1A)] and [**36**, (1.13)].

Now (b) is certainly equivalent to the condition that $\bar{s}$ does not lie in any proper split Levi subgroup of $\bar{\mathbf{G}}^*$. This in turn is equivalent to the condition in (c). $\diamond$

9.4. Examples for the restriction to commutator subgroups

The results above are valid for finite groups of Lie type arising from an algebraic group with connected center. In general, the groups obtained this way are not quasisimple. We therefore have to investigate the descent from a group as above to its commutator subgroup. We include an example, a special case of a much more general example shown to us by Cédric Bonnafé, to give a flavor of the phenomena that can occur.

EXAMPLE 9.6. Let

$$s = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where I_2 denotes the identity matrix of dimension 2. We view s as element of $G = \mathrm{GL}_4(q)$ for some odd q. We then have $C_G(s) = \mathrm{GL}_2(q) \times \mathrm{GL}_2(q)$. Hence $\mathcal{E}(G, [s])$ contains the four characters $\chi_{s,1\otimes 1}$, $\chi_{s,1\otimes q}$, $\chi_{s,q\otimes 1}$, and $\chi_{s,q\otimes q}$, corresponding to the four unipotent characters of $\mathrm{GL}_2(q) \times \mathrm{GL}_2(q)$ (where we have denoted the two unipotent characters of $\mathrm{GL}_2(q)$ by their degrees). Since $C_{\mathbf{G}}(s)$ is a Levi subgroup of $\mathbf{G} = \mathrm{GL}_4(\bar{\mathbb{F}}_q)$, the characters in $\mathcal{E}(G, [s])$ are Harish-Chandra induced from the characters of $\mathcal{E}(C_G(s), [s])$ by Theorem 7.4.

Let z be a generator of $\mathbb{F}_q^* = \mathrm{GL}_1(q)$. We identify the maximally split torus L_0 of G with the direct product $\mathrm{GL}_1(q) \times \cdots \times \mathrm{GL}_1(q)$, so that $L_0 = \{(z^i, z^j, z^k, z^l) \mid 1 \leq i, j, k, l \leq q - 1\}$. Let ψ be the linear complex character of L_0 defined by $\psi(z^i, z^j, z^k, z^l) = (-1)^{k+l}$. Then the Harish-Chandra series of (L_0, ψ) is exactly $\mathcal{E}(G, [s])$. Note that $W_G(L_0, \psi)$ is generated by the two permutation matrices corresponding to the transpositions $(1, 2)$ and $(3, 4)$.

Now let $\tilde{G} = \mathrm{SL}_4(q)$ and $\tilde{L}_0 = \tilde{G} \cap L_0$. The restriction $\tilde{\psi}$ of ψ to $\tilde{L}_0$ is cuspidal and irreducible. However, $W_{\tilde{G}}(\tilde{L}_0, \tilde{\psi})$ is a dihedral group of order 8, since $\tilde{\psi}$ is also invariant under the simultaneous exchange of the first two with the last two coordinates. Thus $R_{\tilde{L}_0}^{\tilde{G}}(\tilde{\psi})$ has exactly five irreducible constituents, one occurring with multiplicity 2. These characters also occur as the irreducible constituents of the restrictions to $\tilde{G}$ of the elements of $\mathcal{E}(G, [s])$, since the restriction of $R_{L_0}^G(\psi)$ to $\tilde{G}$ equals $R_{\tilde{L}_0}^{\tilde{G}}(\tilde{\psi})$ (see, e.g., [**26**, Proposition 13.22]).

It follows from Clifford theory that the two characters $\chi_{s,1\otimes q}$ and $\chi_{s,q\otimes 1}$ of $\mathcal{E}(G, [s])$ restrict to the same irreducible character of $\tilde{G}$, whereas each of the other two elements of $\mathcal{E}(G, [s])$ splits into two conjugate irreducible characters of $\tilde{G}$.

Since only the 2-dimensional irreducible character of $W_{\tilde{G}}(\tilde{L}_0, \tilde{\psi})$ is induced from a proper subgroup, there is at most one irreducible constituent of $R_{\tilde{L}_0}^{\tilde{G}}(\tilde{\psi})$ which is Harish-Chandra imprimitive. In fact, the restriction of $\chi_{s,1\otimes q}$ to $\tilde{G}$ is imprimitive, being Harish-Chandra induced from an irreducible character of degree q of the Levi subgroup $C_{\tilde{G}}(s)$ of $\tilde{G}$. Thus the other four irreducible constituents of $R_{\tilde{L}_0}^{\tilde{G}}(\tilde{\psi})$ are Harish-Chandra primitive (hence primitive by Theorem 6.1).

Let $\bar{s}$ denote the image of s in $\mathrm{PGL}_4(q)$. The irreducible constituents of the restriction of the elements of $\mathcal{E}(G, [s])$ to $\tilde{G}$ are contained in $\mathcal{E}(\tilde{G}, [\bar{s}])$ (see, e.g., [**8**, Proposition 11.7]). Thus, unlike for groups arising from algebraic groups with connected center, a rational Lusztig series can contain primitive and imprimitive characters at the same time.

CHAPTER 10

Exceptional groups

In addition to the classical groups introduced in Section 9.1, we now consider exceptional groups of Lie type. We continue to work with ordinary irreducible characters.

10.1. The exceptional groups of type E and F

Let $\mathbf{G}$ be a simple adjoint algebraic group with a Dynkin diagram of type E or F_4. Moreover, we assume that F is a Frobenius morphism of $\mathbf{G}$ such that, in the respective cases, $G := \mathbf{G}^F$ is a finite group of type $E_i(q)$, $i = 6, 7, 8$, ${}^2E_6(q)$ or $F_4(q)$. In the cases $E_8(q)$ and $F_4(q)$, the group G is simple. In the other cases, the commutator subgroup G' of G is simple.

Since $\mathbf{G}$ is adjoint, its center is trivial and we may thus apply Theorem 8.4. By this and Theorem 7.4, in order to find the Harish-Chandra imprimitive irreducible characters of G, we have to determine the semisimple elements $s \in G^*$ such that $C_{\mathbf{G}^*}(s)$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$. This property only depends on the G^*-conjugacy class of $C_{\mathbf{G}^*}(s)$, hence is a property of the semisimple class type of s. (Two semisimple elements $s_1, s_2 \in G^*$ belong to the same *class type*, if and only if $C_{\mathbf{G}^*}(s)$ and $C_{\mathbf{G}^*}(s)$ are conjugate in G^*; see [**40**, Subsection 4.2].) The semisimple conjugacy classes of the groups considered here have been determined explicitly by Shoji [**103**] and Shinoda [**101**] for type F_4, and by Fleischmann and Janiszczak [**33, 34**] for type E.

The semisimple elements in question can be read off from the tables by Fleischmann and Janiszczak as follows. These tables give, for each semisimple element $s \in G^*$, the order of $Z^\circ(C_{\mathbf{G}^*}(s))^F$ as a polynomial $f_s(q)$ in q. (As in the proof of Theorem 7.6, we write $Z^\circ(C_{\mathbf{G}^*}(s))$ for the connected component of the centre of $C_{\mathbf{G}^*}(s)$.) Then $C_{\mathbf{G}^*}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$, if and only if $f_s(q)$ is divisible by $q-1$ (as a polynomial in q). To see this, first note that the split F-stable Levi subgroups of $\mathbf{G}^*$ are exactly the centralizers of split tori, i.e., F-stable tori of $\mathbf{G}^*$ whose order polynomial is of the form $(x-1)^a$ for some positive integer a (see [**15**, Section 3.D]; for

the concept of order polynomial see [**15**, Définiton 1.9].) If $Z^\circ(C_{\mathbf{G}^*}(s))$ contains a split torus $\mathbf{S}^*$, then $C_{\mathbf{G}^*}(s)$ is contained in the split F-stable Levi subgroup $C_{\mathbf{G}^*}(\mathbf{S}^*)$. On the other hand, if $C_{\mathbf{G}^*}(s)$ is contained in the split F-stable Levi subgroup $\mathbf{L}^*$, then $Z^\circ(\mathbf{L}^*)$ is contained in $Z^\circ(C_{\mathbf{G}^*}(s))$ and so the latter group contains a split torus.

10.2. Explicit results on some exceptional groups

Here we give explicit results on those exceptional groups of Lie type, for which (partial) generic character tables are available in the literature. The following results are all obtained with Theorems 7.4 and 8.4. Occasionally we sketch alternative arguments for some of the results. Note that all groups G considered below can be obtained as $G = \mathbf{G}^F$, where the algebraic group $\mathbf{G}$ satisfies the hypothesis of Section 8.3 (see [**19**, 1.19]).

For the irreducible characters we use the notation of the original sources for the character tables. We also write Φ_m for the value at q of the mth cyclotomic polynomial, $m \in \mathbb{Z}$, $m > 0$.

We begin with the Suzuki groups.

Proposition 10.1. *Let $G = {}^2B_2(q)$ with $q = 2^{2m+1}$, $m \geq 1$ be a simple Suzuki group. Then exactly the irreducible characters of degree q^2+1 are imprimitive. They are Harish-Chandra induced from certain linear characters of the Borel subgroup of G. We have $|\mathrm{Irr}(G)| = q+3$ and $q/2-1$ of these are imprimitive.*

We now consider the simple Ree groups of characteristic 3.

Proposition 10.2. *Let $G = {}^2G_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$ be a simple Ree group (of characteristic 3). Then exactly the irreducible characters of degree q^3+1 are imprimitive. They are Harish-Chandra induced from certain linear characters of the Borel subgroup of G. We have $|\mathrm{Irr}(G)| = q+8$ and $(q-3)/2$ of these are imprimitive.*

We next consider the simple Ree groups of characteristic 2.

Proposition 10.3. *Let $G = {}^2F_4(q)$ with $q = 2^{2m+1}$, $m \geq 1$ be a simple Ree group (of characteristic 2).*

Using the CHEVIE notation for irreducible characters of G, and denoting the maximally split torus of G by T_1, Table 1 lists the imprimitive irreducible characters of G. The column headed "Levi" indicates the smallest Levi subgroup of G of which the irreducible characters are Harish-Chandra induced.

We have $|\mathrm{Irr}(G)| = q^2+4q+17$, and $9/16\,q^2+11/8\,q-5$ of these are imprimitive.

PROOF. We use the notation of Shinoda [**102**] and Malle [**88**] for the maximal tori of G. Four of them, $T_1, \ldots, T_4$ lie in proper Levi subgroups of G. The corresponding irreducible Deligne-Lusztig characters are the characters χ_{28}, χ_{34}, χ_{36}, and χ_{35}, respectively.

Multiplying the trivial character of $\mathrm{SL}_2(q)$ with a non-trivial character of the torus $q-1$ we obtain an irreducible character of the Levi subgroup $\mathrm{SL}_2(q) \times (q-1)$ whose Harish-Chandra induction is a character of type χ_{26}. Similarly, the Steinberg character of $\mathrm{SL}_2(q)$ gives rise to the characters of type χ_{27}. In exactly the same manner we obtain the characters χ_{22}, χ_{25}, χ_{23} and χ_{24} from the unipotent characters of the Suzuki group ${}^2B_2(q)$.

The other irreducible characters of G are primitive, since their degrees do not match any product of the index of a proper parabolic subgroup of G with the degree of an irreducible character of the corresponding Levi subgroup. This has been checked using CHEVIE [**40**]. $\diamond$

The above result is also true for the smallest of the characteristic 2 Ree groups ${}^2F_4(2)$, in the sense that none of its irreducible characters is Harish-Chandra induced. This fact can easily be checked with the Atlas [**21**]. There are of course imprimitive irreducible characters of ${}^2F_4(2)$, whose block stabilizer is the Tits group ${}^2F_4(2)'$. For the latter group see Proposition 5.1.

TABLE 1. Imprimitive irreducible characters of Ree's groups ${}^2F_4(q)$, $q = 2^{2m+1}$

Char.	Degree	Number	Levi
χ_{28}	$\Phi_2^2\Phi_4^2\Phi_6\Phi_{12}$	$\frac{1}{16}(q-2)(q-8)$	T_1
χ_{34}	$\Phi_1\Phi_2\Phi_4^2\Phi_6\Phi_{12}$	$\frac{1}{4}q(q-2)$	$\mathrm{SL}_2(q) \times (q-1)$
χ_{35}	$\Phi_1\Phi_2^2\Phi_6'\Phi_4\Phi_6\Phi_{12}$	$\frac{1}{8}(q-2)(q-\sqrt{2q})$	${}^2B_2(q) \times (q-1)$
χ_{36}	$\Phi_1\Phi_2^2\Phi_6''\Phi_4\Phi_6\Phi_{12}$	$\frac{1}{8}(q-2)(q+\sqrt{2q})$	${}^2B_2(q) \times (q-1)$
χ_{26}	$\Phi_1\Phi_4^2\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	$\mathrm{SL}_2(q) \times (q-1)$
χ_{27}	$q\Phi_1\Phi_4^2\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	$\mathrm{SL}_2(q) \times (q-1)$
χ_{22}	$\Phi_1^2\Phi_4\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	${}^2B_2(q) \times (q-1)$
χ_{25}	$q^2\Phi_1^2\Phi_4\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	${}^2B_2(q) \times (q-1)$
χ_{23}	$\frac{1}{2}\sqrt{2q}\Phi_1\Phi_2^2\Phi_4\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	${}^2B_2(q) \times (q-1)$
χ_{24}	$\frac{1}{2}\sqrt{2q}\Phi_1\Phi_2^2\Phi_4\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	${}^2B_2(q) \times (q-1)$

$$\Phi_6' = q + \sqrt{2q} + 1,\ \Phi_6'' = q - \sqrt{2q} + 1$$

We now investigate the Chevalley groups $G_2(q)$.

PROPOSITION 10.4. *Let* $G = G_2(q)$ *be a simple Chevalley of type* G_2, *where* q *is any prime power larger than* 2.

Using the notation for irreducible characters of G *introduced by Chang and Ree in* [**20**], *Tables 2 and 3 list the imprimitive irreducible characters of* G. *The column headed "Levi" indicates the smallest Levi subgroup of* G *of which the irreducible characters are Harish-Chandra induced.*

If q *is odd and not divisible by* 3, *we have* $|\mathrm{Irr}(G)| = q^2+2q+9$. *For* $q \equiv 1(\mathrm{mod}\ 3)$, *there are* $7/12\,q^2+1/3\,q-71/12$ *imprimitive irreducible characters, whereas for* $q \equiv 2(\mathrm{mod}\ 3)$, *there are* $7/12\,q^2+1/3\,q-17/4$ *of them.*

If q *is a power of* 3, *we have* $|\mathrm{Irr}(G)| = q^2+2q+8$, *and there are* $7/12\,q^2+1/3\,q-17/4$ *imprimitive irreducible characters.*

If q *is even, we have* $|\mathrm{Irr}(G)| = q^2+2q+8$. *If* $q \equiv 1(\mathrm{mod}\ 3)$, *there are* $7/12\,q^2+1/3\,q-14/4$ *imprimitive irreducible characters, wheras if* $q \equiv 2(\mathrm{mod}\ 3)$, *there are* $7/12\,q^2+1/3\,q-3$ *of them.*

PROOF. The irreducible characters of G can be found in [**32**], if q is even, in [**31**], if q is a power of q, and in [**20**], in the general case. Explicit tables are also given in [**46**], as well as in CHEVIE, where a dicitonary is given to compare the various notations in these papers. ◇

TABLE 2. Imprimitive irreducible characters of the Chevalley groups of type $G_2(q)$, q odd

Char.	Degree	Number $q \equiv 1(\mathrm{mod}\ 3)$	Number $q \not\equiv 1(\mathrm{mod}\ 3)$	Levi
X_1	$\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{12}(q^2-8q+19)$	$\frac{1}{12}(q^2-8q+15)$	$(q-1)^2$
X_a, X_b	$\Phi_1\Phi_2\Phi_3\Phi_6$	$\frac{1}{4}(q^2-2q+1)$	$\frac{1}{4}(q^4-2q+1)$	$\mathrm{GL}_2(q)$
X'_{1a}	$\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-5)$	$\frac{1}{2}(q-3)$	$\mathrm{GL}_2(q)$
X_{1a}	$q\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-5)$	$\frac{1}{2}(q-3)$	$\mathrm{GL}_2(q)$
X'_{1b}	$\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-3)$	$\mathrm{GL}_2(q)$
X_{1b}	$q\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-3)$	$\mathrm{GL}_2(q)$

Finally we consider Steinberg's triality groups ${}^3D_4(q)$.

PROPOSITION 10.5. *Let* $G = {}^3D_4(q)$ *be a Steinberg triality group, where* q *is any prime power. Using the notation for irreducible characters of* G *introduced in* [**25**], *Table 4 lists the imprimitive irreducible characters of* G. *The column headed "Levi" indicates the smallest Levi subgroup of* G *of which the irreducible characters are Harish-Chandra induced.*

TABLE 3. Imprimitive irreducible characters of the Chevalley groups of type $G_2(q)$, q even

Char.	Degree	Number $q \equiv 1 (\mathrm{mod}\ 3)$	Number $q \equiv 2 (\mathrm{mod}\ 3)$	Levi
X_1	$\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{12}(q^2-8q+16)$	$\frac{1}{12}(q^2-8q+12)$	$(q-1)^2$
X_a, X_b	$\Phi_1\Phi_2\Phi_3\Phi_6$	$\frac{1}{4}(q^2-2q)$	$\frac{1}{4}(q^4-2q)$	$\mathrm{GL}_2(q)$
X'_{1a}	$\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-4)$	$\frac{1}{2}(q-2)$	$\mathrm{GL}_2(q)$
X_{1a}	$q\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-4)$	$\frac{1}{2}(q-2)$	$\mathrm{GL}_2(q)$
X'_{1b}	$\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-2)$	$\frac{1}{2}(q-2)$	$\mathrm{GL}_2(q)$
X_{1b}	$q\Phi_2\Phi_3\Phi_6$	$\frac{1}{2}(q-2)$	$\frac{1}{2}(q-2)$	$\mathrm{GL}_2(q)$

If q is even, have $|\mathrm{Irr}(G)| = q^4+q^3+q^2+q+5$, and of these,

$$\frac{7}{12}q^4+\frac{1}{6}q^3-\frac{5}{6}q^2-\frac{2}{3}q-3$$

are imprimitive. If q is odd, have $|\mathrm{Irr}(G)| = q^4+q^3+q^2+q+6$, and of these,

$$\frac{7}{12}q^4+\frac{1}{6}q^3-\frac{5}{6}q^2-\frac{2}{3}q-\frac{17}{4}$$

are imprimitive.

PROOF. There are three maximal tori lying in proper Levi subgroups, namely T_0, T_1 and T_2 (in the notation of [**25**]). The corresponding irreducible Deligne-Lusztig characters are the characters named χ_6, χ_8, and χ_{11}, respectively.

The characters of type $\chi_{3,1}$ and $\chi_{3,\mathrm{St}}$ are obtained by Harish-Chandra inducing suitable characters of T_0, the maximally split torus of G. If λ is such a linear character of T_0, we have $R_{T_0}^G(\lambda) = \chi_1 + \chi_{\mathrm{St}}$, with χ_1 of type $\chi_{3,1}$ and χ_{St} of type $\chi_{3,\mathrm{St}}$. If L denotes the Levi subgroup $\mathrm{SL}_2(q^3) \times (q-1)$, then $R_{T_0}^L(\lambda) = \psi_1 + \psi_{\mathrm{St}}$, with ψ_1 of degree 1 and ψ_{St} of degree q^3. It follows that $\chi_1 = R_L^G(\psi_1)$ and $\chi_{\mathrm{St}} = R_L^G(\psi_{\mathrm{St}})$. The assertion for the characters of type $\chi_{5,1}$ and $\chi_{5,\mathrm{St}}$ is proved similarly.

The numbers of the irreducible characters of each type can be found in [**25**].

The other irreducible characters of G are primitive, since their degrees do not match any product of the index of a proper parabolic subgroup of G with the degree of an irreducible character of the corresponding Levi subgroup. This has been checked using CHEVIE [**40**]. $\diamond$

TABLE 4. Imprimitive irreducible characters of Steinberg's triality groups

Char.	Degree	Number q even	Number q odd	Levi
χ_6	$\Phi_2^2\Phi_3\Phi_6^2\Phi_{12}$	$\frac{1}{12}(q^4 - 4q^3 + 2q^2 - 2q + 12)$	$\frac{1}{12}(q^4 - 4q^3 + 2q^2 - 2q + 15)$	$(q^3-1) \times (q-1)$
χ_8	$\Phi_1\Phi_2\Phi_3\Phi_6^2\Phi_{12}$	$\frac{1}{4}(q^4 - 2q)$	$\frac{1}{4}(q^4 - 2q + 1)$	$\mathrm{SL}_2(q) \times (q^3-1)$
χ_{11}	$\Phi_1\Phi_2\Phi_3^2\Phi_6\Phi_{12}$	$\frac{1}{4}(q^4 - 2q^3)$	$\frac{1}{4}(q^4 - 2q^3 + 1)$	$\mathrm{SL}_2(q^3) \times (q-1)$
$\chi_{3,1}$	$\Phi_2\Phi_3\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	$\frac{1}{2}(q-3)$	$\mathrm{SL}_2(q^3) \times (q-1)$
$\chi_{3,\mathrm{St}}$	$q^3\Phi_2\Phi_3\Phi_6\Phi_{12}$	$\frac{1}{2}(q-2)$	$\frac{1}{2}(q-3)$	$\mathrm{SL}_2(q^3) \times (q-1)$
$\chi_{5,1}$	$\Phi_2\Phi_3\Phi_6^2\Phi_{12}$	$\frac{1}{2}(q^3-q^2-q-2)$	$\frac{1}{2}(q^3-q^2-q-3)$	$\mathrm{SL}_2(q) \times (q^3-1)$
$\chi_{5,\mathrm{St}}$	$q\Phi_2\Phi_3\Phi_6^2\Phi_{12}$	$\frac{1}{2}(q^3-q^2-q-2)$	$\frac{1}{2}(q^3-q^2-q-3)$	$\mathrm{SL}_2(q) \times (q^3-1)$

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